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The Theory of Concentrated Langevin Distributions*

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The density of the Langevin (or Fisher–Von Mises) distribution is proportional to $\exp \kappa\mu' x$, where x and the modal vector μ are unit vectors in $\mathbb{R}^q$. κ ($\geqslant 0$) is called the concentration parameter. The distribution of statistics for testing hypotheses about the modal vectors of m distributions simplify greatly as the concentration parameters tend to infinity. The non-null distributions are obtained for statistics appropriate when $\kappa_1,\ldots,\kappa_m$ are known but tend to infinity, and are unknown but equal to κ which tends to infinity. The three null hypotheses are

$$H_{01}: \mu = \mu_0 (m = 1), \qquad H_{02}: \mu_1 = \cdots = \mu_m, \qquad H_{03}: \mu_i \in V, \qquad i = 1,\ldots, m.$$

In each case a sequence of alternatives is taken tending to the null hypothesis.

1. INTRODUCTION

A random vector x in $\mathbb{R}^q$ of length $\|x\|$ unity is said to have the Langevin distribution if its probability density is given by

$$a_q(\kappa)^{-1} \exp \kappa x'\mu \qquad (\|\mu\| = 1, \kappa \geqslant 0) \tag{1.1}$$

on the surface $\Omega_q = \{x \mid \|x\| = 1\}$ with area $\omega_q = 2\pi^{q/2}/\Gamma(q/2)$. The mean or modal direction is called μ, and κ the concentration parameter of (1.1). Writing $x = t\mu + (1-t^2)^{1/2}\,\xi$ with $\|\xi\| = 1$, $\mu'\xi = 0$, $t = x'\mu$, we have

$$d\omega_q = (1-t^2)^{(q-3)/2}\, d\omega_{q-1}, \qquad a_q(\kappa) = (2\pi)^{q/2}\, I_{q/2-1}(\kappa)\, \kappa^{-q/2+1}, \tag{1.2}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind, of order ν.

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Statistical theory for the Langevin distribution was observed, in Watson [3] for $q = 3$, and Watson and Williams [4], for general q, to become much simpler when $\kappa \to \infty$. The proofs when given were informal or used limits of exact results. An account of the theory for fixed q, n, and κ with remarks about some optimal tests has been given by Mardia [1]. Watson [2] gave a self-contained account of estimation and testing for all q and κ as the sample sizes n_i tend to infinity. That paper gives, for the first time, tests for relations between the modal vectors of populations with possibly different concentration κ_i and the non-null distributions of all the test statistics proposed.

In Section 2 we find the non-null distributions of

$$2\kappa(\|X\| - \mu_0' X) \qquad (H_{01}: \mu = \mu_0), \tag{1.3}$$

$$2\left(\sum_{i=1}^{m} \kappa_i \|X_i\| - \left\|\sum_{i=1}^{m} \kappa_i X_i\right\|\right) \qquad (H_{02}: \mu_i = \mu_j, \forall i,j), \tag{1.4}$$

$$2\sum_{i=1}^{m} \kappa_i(\|X_i\| - \|X_{iV}\|) \qquad (H_{03}: \mu_i \in V, \forall i), \tag{1.5}$$

when $\kappa, \kappa_1,..., \kappa_m \to \infty$ and the alternative hypotheses tend to H_0 at rate $\kappa^{-1/2}$. In (1.5) X_{iV} is the part of X_i in the subspace V. The sample sizes $n, n_1,..., n_m$ will be fixed. These are some of the test statistics studied in Watson [2] where the n's $\to \infty$ and the κ's are fixed. No claims are made for the optimality of these statistics here, but they each have geometric forms which make them suitably invariant, appropriate to their associated null hypotheses, and simply distributed as the concentrations tend to infinity.

In Section 3, we derive the non-null F distributions of the Watson and Williams F statistics for testing H_{01}, H_{02}, and H_{03} when the κ_i are equal but unknown. These statistics are obtained by replacing the κ_i's in (1.3), (1.4), and (1.5) by an estimate of the common κ, and are stated explicitly in (3.9), (3.11), and (3.12).

In Section 4, some inconclusive remarks are made about the statistics that might be used if the κ_i are unknown and unequal.

2. Tests When the Concentration arameters Are Known and Large

(A) It follows from (1.1), (1.2), and the asymptotic expansion for $I_\nu(\kappa)$, that the parts of x along μ, $\mu' x$ and orthogonal to μ, $x_{\perp\mu}$ are independent, and that, as $\kappa \to \infty$, $2\kappa(1 - \mu' x)$ is asymptotically X^2_{q-1}, and that $\kappa^{1/2} x_{\perp\mu}$ is asymptotically Gaussian with zero mean vector and covariance matrix $P_\mu = I - \mu\mu'$. Briefly $\mathscr{L}\kappa^{1/2} x_{\perp\mu} \to G_q(0, P_\mu)$.

(B) To study the test (1.3) we consider a sequence of alternatives with

$$\begin{aligned}\mu &= (\mu_0 + \kappa^{-1/2}\delta)(1 + \kappa^{-1}\delta'\delta)^{-1/2}, \qquad \delta \perp \mu_0 \\ &= \mu_0 + \kappa^{-1/2}\delta - \kappa^{-1}\frac{\delta'\delta}{2} + O(\kappa^{-3/2}).\end{aligned} \tag{2.1}$$

If we now write

$$x = \mu_0 + (1-u^2)^{1/2}\,\eta, \qquad \eta \perp \mu_0, \quad \eta'\eta = 1, \tag{2.2}$$

the joint density of u and η is seen to be proportional to

$$\exp(-\kappa(1-u) + \kappa^{1/2}(1-u^2)^{1/2}\,\delta'\eta - \tfrac{1}{2}\,\delta'\delta u)(1-u^2)^{(q-3)/2} \tag{2.3}$$

on neglecting terms $O(\kappa^{-1/2})$. Hence if we set $v = 2\kappa(1-u)$ and let $\kappa \to \infty$, (2.3) becomes

$$\exp\left(-\frac{v}{2} + v^{1/2}\delta'\eta - \frac{\delta'\delta}{2}\right) v^{((q-1)/2)-1}. \tag{2.4}$$

This shows that

$$\mathscr{L}(w = v^{1/2}\eta) \to G_q(\delta, P_{\mu_0}), \tag{2.5}$$

so that

$$\mathscr{L} w'w \to X^2_{q-1}(\delta'\delta). \tag{2.6}$$

If $x_1, x_2, \ldots, x_n$ are independent observations from (1.1), $X = \sum_1^n x_i$ is sufficient for κ and μ. Since (2.2) can be written as

$$\begin{aligned}x &= \mu_0\left(1 - \frac{v}{2\kappa}\right) + \kappa^{-1/2}w + O(\kappa^{-3/2}), \\ X &= \mu_0 \sum_1^n \left(1 - \frac{v_i}{2\kappa}\right) + \kappa^{-1/2}\Sigma w_i + O(\kappa^{-3/2}).\end{aligned} \tag{2.7}$$

The remainder in (2.7) is of order $\kappa^{-3/2}$ in probability. Thus

$$X'X = n^2 - \frac{1}{2\kappa}\sum_{i,j}(v_i + v_j - w_i'w_j - w_j'w_i) + O(\kappa^{-2})$$

or

$$X'X = n^2 - \frac{n}{\kappa}\sum_1^n (w_i - \bar{w})'(w_i - \bar{w}) + O(\kappa^{-2}), \tag{2.8}$$

where $\bar{w} = n^{-1}\Sigma w_i$. From (2.7), $X \to n\mu_0$, $\|X\| \to n$, in probability as $\kappa \to \infty$, and

$$\mu_0' X = \sum_1^n (1 - v_i/2\kappa) + O(\kappa^{-3/2}), \tag{2.9}$$

and

$$X'X = (\mu_0 X)^2 + \frac{n^2}{\kappa} \bar{w}^2 + O(\kappa^{-2}). \tag{2.10}$$

Thus from (2.8), taking $\|X\|^2 = X'X$,

$$\kappa(n^2 - \|X\|^2) = n \sum (w_i - \bar{w})'(w_i - \bar{w}) + O(\kappa^{-1}), \tag{2.11}$$

$$\kappa(\|X\|^2 - (\mu_0' X)^2) = n^2 \bar{w}' \bar{w} + O(\kappa^{-1}), \tag{2.12}$$

so that

$$\kappa(n^2 - (\mu_0' X)^2) = n \sum w_i' w_i + O(\kappa^{-1}). \tag{2.13}$$

Equations (2.11), (2.12), and (2.5) show that $2\kappa(n - \|X\|)$ and $2(\|X\| - \mu_0' X)$ are, as $\kappa \to \infty$, asymptotically independent non-central chi-squares with degrees of freedom, respectively $(n-1)(q-1)$ and $(q-1)$, and non-centrality parameters zero and $n\|\delta\|^2$, respectively. The latter result gives the non-null distribution of the statistic (1.3) which is asymptotically equivalent to $\kappa\|X_{\perp\mu 0}\|^2/n$.

(C) We study the statistic (1.4) given a sample of n_i from the ith of m populations of the form (1.1) with modal vectors

$$\mu_i = (\mu_0 + \kappa_i^{-1/2}\delta_i)(1 + \kappa_i^{-1}\delta_i'\delta_i)^{-1/2}, \qquad \delta_i \perp \mu_0, \tag{2.14}$$

and concentration parameters κ_i. The ith sample vector resultant may be written, as in (2.7),

$$X_i = \mu_0 \sum_{j=i}^{n_i} \left(1 - \frac{v_{ij}}{2\kappa_i}\right) + \kappa_i^{-1/2} \sum_{j=1}^{n_i} w_{ij} + O(\kappa_i^{-1/2}). \tag{2.15}$$

Thus

$$\sum_1^m \kappa X_i = \mu_0 \left(\sum_{i=1}^m \kappa_i n_i - \tfrac{1}{2} \sum_i \sum_j v_{ij}\right) + \sum_{i=1}^m \kappa_i^{1/2} \sum_{i=1}^{n_i} w_{ij} + O(\min_i \kappa_i)^{-1/2}. \tag{2.16}$$

We will suppose that

$$\lambda_i = \lim_{\kappa_i\text{'s}\to\infty} \frac{\kappa_i n_i}{\sum \kappa_i n_i} > 0 \tag{2.17}$$

so that $\sum \lambda_i = 1$.

From (2.16), and recalling that $v_{ij} = w_{ij}' w_{ij}$,

$$\frac{\|\sum \kappa_i X_i\|}{\sum \kappa_i n_i} \to 1 \qquad \text{(prob)}, \tag{2.18}$$

and

$$\frac{(\sum \kappa_i n_i)^2 - \|\sum \kappa_i X_i\|^2}{\sum \kappa_i n_i} = \sum\sum v_{ij} - \frac{(\sum \kappa_i^{1/2} \sum w_{ij})'(\sum \kappa_i^{1/2} w_{ij})}{\sum \kappa_i n_i} + O\left(\min \frac{\lambda_i}{\kappa_i}\right), \tag{2.19}$$

so that

$$2\left(\sum_1^m \kappa_i n_i - \left\|\sum \kappa_i X_i\right\|\right) = \sum\sum w_{ij}' w_{ij} - \frac{(\sum \kappa_i^{1/2} n_i \bar{w}_i)'(\sum \kappa_i^{1/2} n_i \bar{w}_i)}{\sum \kappa_i n_i} + O\left(\min \frac{\lambda_i}{\kappa_i}\right). \tag{2.20}$$

But from (2.11)

$$2\left(\sum \kappa_i n_i - \sum \kappa_i \|X_i\|\right) = \sum_i \sum_j (w_{ij} - \bar{w}_i)'(w_{ij} - \bar{w}_i) + O(\min \kappa_i)^{-1}, \tag{2.21}$$

so subtracting (2.21) from (2.20) and letting the κ_i's $\to \infty$,

$$2\left(\sum \kappa_i \|X_i\| - \left\|\sum \kappa_i X_i\right\|\right) \underset{d}{\to} \sum n_i \bar{w}_i' \bar{w}_i - \frac{\|\sum \kappa_i^{1/2} n_i \bar{w}_i\|^2}{\sum \kappa_i n_i}, \tag{2.22}$$

where the $\bar{w}_i$ are independent and $G_q(\delta_i, n_i^{-1} P_{\mu_0})$ by (2.5). Hence, as $\kappa \to \infty$,

$$\mathscr{L} 2\left(\sum \kappa_i \|X_i\| - \left\|\sum \kappa_i X_i\right\|\right) \to \chi^2_{(m-1)(q-1)}(\lambda),$$
$$\lambda = \sum n_i \|\delta_i\|^2 - \left\|\sum (n_i \lambda_i)^{1/2} \delta_i\right\|^2, \tag{2.23}$$

which gives the non-null distribution of the test statistics (1.4). For use in the next section we note that $\sum \kappa_i(n_i - \|X_i\|)$ and $\sum \kappa_i \|X_i\| - \|\sum \kappa_i X_i\|$ are asymptotically independent, since the former depends on deviations from means, the latter, only on means. Furthermore (2.21) also shows that, as the κ_i's $\to \infty$,

$$\mathscr{L}2\left(\sum \kappa_i(n_i - \|X_i\|)\right) \to \chi^2_{(n-m)(q-1)}, \tag{2.24}$$

where $n = \sum n_i$.

(D) We study the statistic (1.5) for the sequence of alternatives $(i = 1,..., m)$,

$$\mu_{i1} = (\mu_{i0} + \kappa_i^{-1/2}\delta_i)(1 + \kappa_i^{-1}\delta_i'\delta_i)^{-1/2}, \qquad \mu_{i0} \in \mathscr{V}, \quad \delta_i \in \mathscr{V}^{\perp} \tag{2.25}$$

so that

$$\mu_{i1} = \mu_{i0} + \kappa_i^{-1/2}\,\delta_i - \frac{1}{2\kappa_i}\,\delta_i'\,\delta_i\mu_{i0} + O(\kappa_i^{-1/2}). \tag{2.26}$$

Here $\mathscr{V}$ is a subspace of dimension $q - s$, and $\mathscr{V}^{\perp}$ its orthogonal complement. Let P_v and $P_{v\perp}$ be the orthogonal projectors onto these subspaces so that $X_{iv} = P_v X_i$. Hence from (2.15),

$$X_{iv} = \mu_{i0} \sum_j \left(1 - \frac{v_{ij}}{2\kappa}\right) + \kappa^{-1/2} n_i P_v \bar{w}_i + O(\kappa_i^{-3/2}) \tag{2.27}$$

so that X_{iv}, like X_i, tends to $n_i\mu_{i0}$ and $X_{iv\perp}$ tends to the zero vector in probability as $\kappa \to \infty$. Thus

$$\|X_{iv}\|^2 = n_i^2 - \frac{n_i}{\kappa_i}\sum_j v_{ij} + \frac{n_i^2}{\kappa_i}\|P_v\bar{w}_i\|^2 + O(\kappa_i^{-3/2}),$$

so that

$$\begin{aligned} 2\kappa_i(n_i - \|X_{iv}\|) &= \sum_j w_{ij}'w_{ij} - n_i\|P_v\bar{w}_i\|^2 + O(\kappa_i^{-1/2}) \\ &= \sum_j (w_{ij} - \bar{w}_i)'(w_{ij} - \bar{w}_i) + n_i\|P_{v\perp}\bar{w}_i\|^2 + O(\kappa_i^{-1/2}). \end{aligned} \tag{2.28}$$

Also

$$\kappa_i(\|X_i\|^2 - \|X_{iv}\|^2) = \kappa_i\,\|P_{v\perp}X_i\|^2,$$

so that

$$2\kappa_i(\|X_i\| - \|X_{iv}\|) = n_i\,\|P_{v\perp}\bar{w}_i\|^2 + O(\kappa_i^{-2}). \tag{2.29}$$

Subtracting (2.29) from (2.28), we obtain

$$2\kappa_i(n_i - \|X_i\|) = \sum_j (w_{ij} - \bar{w}_i)(w_{ij} - \bar{w}_i) + O(\kappa_i^{-1/2}) \tag{2.30}$$

Equations (2.29) and (2.30) show that $2\kappa_i(\|X_i\| - \|X_{iv}\|)$ and $2\kappa_i(n_i - \|X_i\|)$ are asymptotically independent chi-squares with degrees of freedom s and $(n_i - 1)(q-1)$, and non-centrality parameters $n_i \|\delta_i\|^2$ and zero, respectively. Thus the distribution of (1.5), $2\sum_{i=1}^{m} \kappa_i(\|X_i\| - \|X_{iv}\|)$ is $\chi^2_{ms}(\sum n_i \|\delta_i\|^2)$.

3. Tests When the Large Concentration Parameters Are Only Known to Be Equal

When all the $n_i \to \infty$, consistent estimates of the κ_i's may replace them in the statistics (1.3), (1.4), (1.5) to give tests suitable for unknown concentrations. Further, the large sample behavior of the statistics is unchanged by these substitutions. This strategy is *not* available when the n_i are fixed but it is known that all the κ_i are large.

Writing $A_q(\kappa) = a_q'(\kappa)/a_q(\kappa)$, the maximum likelihood estimates of κ, given a sample of n, are defined by

$$A_q(\hat{\kappa}_\mu) = \mu' X/n \qquad (\mu \text{ known}), \tag{3.1}$$

$$A_q(\hat{\kappa}) = \|X\|/n \qquad (\mu \text{ unknown}). \tag{3.2}$$

In Watson [2], for example, it is shown that

$$\begin{gathered} 0 \leqslant A_q(\kappa) \leqslant 1, \qquad A_q'(\kappa) > 0, \\ A_q(\kappa) = 1 - \frac{(q-1)}{2}\frac{1}{\kappa} + \frac{(q-1)(q-3)}{8}\frac{1}{\kappa^2} + O\left(\frac{1}{\kappa^3}\right). \end{gathered} \tag{3.3}$$

From (3.3) we see that (3.1) and (3.2) always have unique solutions. If these solutions are large, (3.4) shows that they will be, to a first approximation,

$$\hat{\kappa}_\mu = \frac{(q-1)}{2}\frac{n}{n - \mu' X}, \qquad \hat{\kappa} = \frac{(q-1)}{2}\frac{n}{n - \|X\|}. \tag{3.4}$$

The limiting distributional statements are

$$2n\kappa(1 - A_q(\hat{\kappa}_\mu)) \underset{d}{\rightarrow} \chi^2_{n(q-1)}, \qquad 2n\kappa(1 - A_q(\hat{\kappa})) \underset{d}{\rightarrow} \chi^2_{(n-1)(q-1)} \qquad (\kappa \to \infty). \tag{3.5}$$

These statements only show that $\hat{\kappa}_\mu$ and $\hat{\kappa}$ tend to infinity. It is not true that they tend to κ. However, it is true that

$$\frac{\hat{\kappa}}{\kappa} \underset{d}{\rightarrow} \frac{n(q-1)}{\chi^2_{(n-1)(q-1)}}, \qquad \frac{\hat{\kappa}_\mu}{\kappa} \underset{d}{\rightarrow} \frac{n(q-1)}{\chi^2_{n(q-1)}}. \tag{3.6}$$

The last assertions provide a simple way of making tests and finding confidence intervals for κ's as suggested by Watson and Williams [4].

To make tests of relations between modal vectors when the concentration parameters are known to be equal to κ, say, we may proceed as follows:

$$2\hat{\kappa}(\|X\| - \mu_0' X) \underset{d}{\rightarrow} n(q-1)\frac{(\|X\| - \mu_0' X)}{n - \|X\|}, \tag{3.7}$$

which is proportional to

$$\frac{(\|X\| - \mu_0' X)/(q-1)}{(n - \|X\|)/(n-1)(q-1)} \underset{d}{\rightarrow} F_{q-1,(n-1)(q-1)}, \tag{3.8}$$

when the null hypothesis H_{01} in (1.3) s true, by the results of Section 3(B).

The statistic (1.4) becomes $2\kappa(\sum \|X_i\| - \|\sum X_i\|)$. A pooled estimator is given by $\hat{\kappa} = n(q-1)/2(n - \sum \|X_i\|)$, $n = \sum n_i$. We are led to a test statistic proportional to

$$\frac{\sum \|X_i\| - \|\sum X_i\|}{\sum (n_i - \|X_i\|)}. \tag{3.9}$$

By the results of Section 3(C), we may use

$$\frac{(\sum \|X_i\| - \|\sum X_i\|)/(m-1)(q-1)}{(n - \sum \|X_i\|)/(n-m)(q-1)} \underset{d}{\rightarrow} F_{(m-1)(q-1),(n-m)(q-1)} \qquad (\kappa \rightarrow \infty), \tag{3.10}$$

when H_{02} is true.

The statistic (1.5) becomes $2\kappa \sum_{i=1}^{m} (\|X_i\| - \|X_{iV}\|)$, and we will replace κ with $\hat{\kappa}$, so the new statistic is proportional to

$$\frac{\sum (\|X_i\| - \|X_{iV}\|)}{\sum (n_i - \|X_i\|)}.$$

Suppose that the subspace V has dimension $q - s$ and hence $\dim(V^{\perp}) = s$. It then follows from the results of Section 3(D) that

$$\frac{\sum (\|X_i\| - \|X_{iV}\|)/ms}{(n - \sum \|X_i\|)/(n-m)(q-1)} \underset{d}{\rightarrow} F_{ms,(n-m)(q-1)}, \tag{3.11}$$

when H_{03} is true.

The results (3.8), (3.10), (3.11) justify the tests proposed by Watson and Williams [4] when all populations have the *same* concentration parameter. In these circumstances we can give the non-null distributions of these test statistics (and hence compute their powers) by using the results of Section 2. We will use the same sequences of alternatives as in Section 2. They will be non-central F distributions.

4. Tests When the Concentrations Are Large but Possibly Different

Here we must adapt the statistics $2(\sum \kappa_i \|X_i\| - \|\sum \kappa_i X_i\|)$ and $2\sum \kappa_i(\|X_i\| - \|X_{iv}\|)$ for unknown and unequal κ_i's. Replacing the κ_i's by their estimators as in the last section, the former becomes proportional to

$$\sum_1^m \frac{n_i \|X_i\|}{n_i - \|X_i\|} - \left\| \sum_1^m \frac{n_i X_i}{n_i - \|X_i\|} \right\|, \tag{4.1}$$

the latter to

$$\sum_1^m \frac{n_i}{n_i - \|X_i\|} (\|X_i\| - \|X_{iv}\|). \tag{4.2}$$

As all the $\kappa_i \to \infty$, it is clear that (4.2) becomes a linear function of independent non-central F's, a distribution which is not tabulated. The statistic (4.1) becomes a more complicated function on non-central F's. Thus neither of these statistics seem to have a distribution that will be useful in applications at the moment.

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