A TEST FOR RANDOMNESS OF DIRECTIONS

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Summary

The 5 per cent and 1 per cent significance points are given for a test of randomness of unit vectors in three dimensions. The test has been designed for use in the analysis of palaeomagnetic data.

Introduction.—In many problems a test of randomness of the direction of a vector is required. For example, in palaeomagnetism the direction of remanent magnetism of specimens from a given site has been supposed by Fisher (1953) to follow the probability distribution

$$c \exp(\kappa \cos\theta),$$

where θ is the angle between the observed direction and the mean direction and κ is a precision constant.

Watson (1956) has given significance tests for various hypotheses concerning κ and the mean direction; in particular a test of $\kappa=0$ was given. Since, when κ equals zero, this density function is constant, the null-hypothesis $\kappa=0$ is the hypothesis of randomness. The purpose of the present paper is to give a short table of significance points to facilitate the practical application of the previously suggested test for randomness. While the test is optimal when the density suggested by Fisher is the alternative to randomness, it is a valid test no matter what the alternative density may be.

Construction of the test.—If R is the length of the resultant of N randomly directed unit vectors, Fisher (*loc. cit.*) has shown that

$$P(R>R_0)=\int_{R_0}^{N} \frac{R\phi_N(R)dR}{2^{N-1}}, \tag{1}$$

where

$$\phi_N(R)=\frac{1}{(N-2)!}\sum_{r=0}^{\infty}(-1)^r\binom{N}{r}\langle N-R-2r\rangle^{N-2}, \tag{2}$$

with the notation $\langle x\rangle=x$ if $x\geqslant 0$, $\langle x\rangle=0$ if $x\leqslant 0$.

It may be shown that

$$P(R>R_0)=\alpha_0(R_0)-\alpha_1(R_0)+\alpha_2(R_0)-\cdots, \tag{3}$$

where

$$\alpha_r(R_0)=\binom{N}{r}\frac{1}{(N-2)!}\left\langle\frac{N-R_0-2r}{2}\right\rangle^{N-1}\left\{\frac{N-2r}{N-1}-\frac{N-R_0-2r}{N}\right\}. \tag{4}$$

Rayleigh (1919) derived an asymptotic expansion for the probability density function of R. Writing $y=3R^2/2N$, he found as the density function of y,

$$f(y)=\frac{y^{1/2}e^{-y}}{\Gamma(3/2)}+\frac{3}{4N}\left(-\frac{y^{1/2}e^{-y}}{\Gamma(3/2)}+2\frac{y^{3/2}e^{-y}}{\Gamma(5/2)}-\frac{y^{5/2}e^{-y}}{\Gamma(7/2)}\right)+O\left(\frac{1}{N^2}\right). \tag{5}$$

The leading term shows us that R is approximately distributed as $\sqrt{N\chi_3^2/3}$, where χ_3^2 stands for a chi-square variate with three degrees of freedom.

To define a significance test, we find, for each N, a value of R that will be exceeded, on the hypothesis of randomness, with a specified probability α. If the observed value of R exceeds R_0, we agree to reject the null-hypothesis of randomness. If R is less than R_0, we say that the null-hypothesis is not disproved. Values of α equal to 0·05 or 0·01 are often used for this purpose. The appropriate value of R_0 is called a significance point.

Using (3) and (4), it is possible, by inverse interpolation, to find the required values of R_0 exactly. For N above 20, the arithmetic labour soon becomes excessive and one is forced to use Rayleigh's approximation. The 5 per cent and 1 per cent significance points given below were found from (3) and (4). For $N \geqslant 20$, 5 per cent significance points correct to two decimal places may be found by using (5) with Pearson's (1951) tables of the incomplete gamma-function. It was however observed that for $20 \leqslant N \leqslant 100$, the approximation $\sqrt{N\chi_3^2/3}$, where $\chi_3^2 = 7{\cdot}815$, was always 0·04 above the true value. For $N \geqslant 20$, 1 per cent significance points may be found using (5); the values so obtained are correct to two decimal places if $N \geqslant 30$, and 0·01 above the correct value if $20 \leqslant N < 30$. The approximation $\sqrt{N\chi_3^2/3}$, where $\chi_3^2 = 11{\cdot}345$, gives values which are never more than 0·15 above the correct value for $N \geqslant 20$.

Significance points of R

Sample size N	5 per cent significance point	1 per cent significance point
5	3·50	4·02
6	3·85	4·48
7	4·18	4·89
8	4·48	5·26
9	4·76	5·61
10	5·03	5·94
11	5·29	6·25
12	5·52	6·55
13	5·75	6·84
14	5·98	7·11
15	6·19	7·36
16	6·40	7·60
17	6·60	7·84
18	6·79	8·08
19	6·98	8·33
20	7·17	8·55

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