Geophys. J. Int. (1992) **109**, 225–232

Testing whether two Fisher distributions have the same centre

Geoffrey S. Watson[1] and Michel G. Debiche[2]

[1] *Department of Statistics, Princeton University, Princeton, NJ 08544, USA*

[2] *Department of Geological & Geophysical Sciences, Princeton University, Princeton, NJ 08544, USA*

Accepted 1991 October 30. Received 1991 October 30; in original form 1991 April 12

SUMMARY

We computed by simulation the null distributions and power function of 10 statistics for comparing the mean directions of two Fisher distributions for various sample sizes and concentrations. 10 000 simulations were used for each null distribution and for each degree of angular separation out to 40°. The statistics have been suggested by, or are variations of suggestions of, McFadden & Lowes (1981) (which we show are not conditional as they insisted) and Watson (1956, 1983).

Key words: concentrations, Fisher distribution, mean directions, significance tests.

1 INTRODUCTION

We suppose that, with $i = 1, 2$, a sample of size N_i has been drawn from a Fisher distribution with concentration κ_i and centre or mean direction $\boldsymbol{\mu}_i$. With this data we wish to test that $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$. We will use the original notation and results of Fisher (1953). All statistics must be based upon the sufficient statistics $\mathbf{R}_1$ and $\mathbf{R}_2$, the sample sums or resultants, since they contain all the information in the data about the unknown parameters. The concentrations may be known or unknown; hence our statistics come in pairs and we have chosen their designations accordingly, e.g. a & b, c_{12} & $\hat{c}_{12}$, g & h, t'_θ & $t_{\hat\theta}$, u & v. A different rule relates f and f_{mc}. We will now derive approximate distributions for all plausible statistics known to us. This simultaneously summarizes the literature.

The likelihood L of a sample of size N is

$$L = (\kappa/4\pi \sinh \kappa)^N \exp(\kappa\boldsymbol{\mu}\cdot\mathbf{R}) \tag{1}$$

where $\boldsymbol{\mu}\cdot\mathbf{R}$ denotes the scalar product of the two vectors $\boldsymbol{\mu}$ and $\mathbf{R}$, and so is maximized when

$$\hat{\boldsymbol{\mu}} = \mathbf{R}/\|\mathbf{R}\| \quad \text{and} \quad \coth k - 1/k = R/N, \tag{2}$$

where $\hat{\boldsymbol{\mu}}$ and k are estimators of $\boldsymbol{\mu}$ and κ, respectively, and $\|\mathbf{R}\|$ and R both denote the length of the vector $\mathbf{R}$. In the fields of application we have in mind, κ is large (say 20 or more) and N is sufficiently large so that, instead of equation (2), we will be able to use the conventional estimator from Fisher (1953),

$$k = (N-1)/(N-R). \tag{3}$$

For a brief review of the literature on the estimation of κ as well as general numerical background see Fisher, Lewis & Embleton (1987). For a simulation study of estimators of κ in the range of values relevant to this paper see Debiche & Watson (1991). Writing $\boldsymbol{\mu}\cdot\mathbf{R} = R\cos\theta$, the joint distribution of R and θ obtained by integrating equation (1) over all samples with given R and θ must have the form

$$g(R)(\kappa/4\pi \sinh \kappa)^N \exp(\kappa R\cos\theta)\sin\theta. \tag{4}$$

The term $g(R)\sin\theta(4\pi)^{-N}$ forms the joint density of R and θ when $\kappa = 0$ and the Fisher parental distribution is simply the uniform distribution on the sphere. The function $g(R)$ is known but its specific form will not be used here. The marginal distribution of R is the integral of equation (4) with respect to θ. Thus the *conditional distribution of θ, given R*, being equation (4) divided by this integral, must be proportional to $\sin\theta \exp(\kappa R\cos\theta)$. After normalization we obtain the density

$$[(\kappa R)/2\sinh(\kappa R)]\exp(\kappa R\cos\theta)\sin\theta, \tag{5}$$

as the conditional density of θ, given R. This Fisher (1953) result is the basis of the suggestions of McFadden & Lowes (1981) outlined below.

The argument of McFadden & Lowes (1981) is based on the following approximations. If, as is usual in our applications, κR is large, θ will be small and so equation (5) implies that

$$\kappa R\theta^2 \sim \chi_2^2, \tag{6}$$

where $\sim$ stands for 'is distributed as' and χ_2^2 stands for the chi-square distribution with two degrees of freedom. Suppose θ and ϕ are the spherical polar angles of the direction $\hat{\boldsymbol{\mu}}$ of $\mathbf{R}$ when the pole of the coordinate system is $\boldsymbol{\mu}$. From the rotational symmetry of equation (1),

$$\phi \sim \text{uniform on } [0, 2\pi]. \tag{7}$$

If a line from the origin through $\hat{\boldsymbol{\mu}}$ hits the tangent plane (to the sphere at $\boldsymbol{\mu}$) at $\mathbf{P}$ the point (x, y), then x and y are independently normally distributed with mean zero and variance $1/(\kappa R)$, that is,

$$x \quad \text{and} \quad y \sim \mathcal{N}[0, 1/(\kappa R)]. \tag{8}$$

Now return to our two samples whose mean directions $\hat{\boldsymbol{\mu}}_1$ and $\hat{\boldsymbol{\mu}}_2$ lead to two points $\mathbf{P}_1 = (x_1, y_1)$ and $\mathbf{P}_2 = (x_2, y_2)$. Call γ the angular separation of $\hat{\boldsymbol{\mu}}_1$ and $\hat{\boldsymbol{\mu}}_2$. Then because $\kappa_1 R_1$ and $\kappa_2 R_2$ are both large, and because the null hypothesis $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ is supposed to be true, the spherical distance from $\hat{\boldsymbol{\mu}}_1$ to $\hat{\boldsymbol{\mu}}_2$ will be roughly the distance between $\mathbf{P}_1$ and $\mathbf{P}_2$ so that

$$\gamma^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2. \tag{9}$$

Defining

$$\sigma^2 \equiv (\kappa_1 R_1)^{-1} + (\kappa_2 R_2)^{-1} \equiv \kappa'^{-1}, \tag{10}$$

it follows that, *for fixed* R_1 *and* R_2,

$$\kappa' \gamma^2 = (\gamma/\sigma)^2 \sim \chi_2^2. \tag{11}$$

By considering the triangle with sides $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2$ when γ is small, McFadden & Lowes (1981) find

$$\gamma^2 = (R_1 R_2)^{-1}[(R_1 + R_2)^2 - R^2]. \tag{12}$$

Combining the last three equations gives their result: conditional upon the fixed values of R_1 and R_2, it is approximately true that

$$a \equiv \kappa_1 \kappa_2 [(R_1 + R_2)^2 - R^2]/[\kappa_1 R_1 + \kappa_2 R_2] \sim \chi_2^2. \tag{13}$$

Notice that the (approximate) conditional distribution, χ_2^2, does not depend upon the values of the conditioners, R_1 and R_2. What McFadden & Lowes (1981) do not observe (in fact they insist on their result being conditional) is that χ_2^2 must therefore be the *unconditional* distribution of their statistic (we will call it a) on the left-hand side of equation (13). We will later examine these approximations by simulation. Since often the κ's will be unknown, it would be natural to replace them by their estimates k_1 and k_2. This leads to the statistic and approximation

$$b \equiv k_1 k_2 [(R_1 + R_2)^2 - R^2]/[k_1 R_1 + k_2 R_2] \sim \chi_2^2. \tag{14}$$

If $\kappa_1 = \kappa_2$, McFadden & Lowes (1981) suggest the specialization of equation (13)

$$c_{12} \equiv \kappa[(R_1 + R_2)^2 - R^2]/[R_1 + R_2] \sim \chi_2^2, \tag{13$'$}$$

which in turn suggests

$$\hat{c}_{12} \equiv k[(R_1 + R_2)^2 - R^2]/[R_1 + R_2] \sim \chi_2^2. \tag{14$'$}$$

The authors then proceed to create an F statistic by using the Watson (1956) approximate result that

$$2\kappa_1(N_1 - R_1) + 2\kappa_2(N_2 - R_2) \sim \chi^2_{2(N-2)}, \tag{15}$$

where $N \equiv N_1 + N_2$. By assuming (they give no proof) that the statistic a in (13) is statistically independent of that in (15), it then follows that

$$g \equiv \frac{\{\kappa_1 \kappa_2 [(R_1 + R_2)^2 - R^2]/(\kappa_1 R_1 + \kappa_2 R_2)\}/2}{[2\kappa_1(N_1 - R_1) + 2\kappa_2(N_2 - R_2)]/[2(N-2)]} \sim F_{2,2(N-2)}, \tag{16}$$

which is equation (23) in their paper. Here $F_{2,2(N-2)}$ stands for the F distribution with 2 and $2(N-2)$ degrees of freedom. We will also define

$$h \equiv \frac{k_1 k_2 [(R_1 + R_2)^2 - R^2]/(k_1 R_1 + k_2 R_2)}{[2k_1(N_1 - R_1) + 2k_2(N_2 - R_2)]/(N-2)} \sim F_{2,2(N-2)}, \tag{17}$$

Finally McFadden & Lowes remark that if $\kappa_1 = \kappa_2$, (15) can be written

$$2\kappa(N - R_1 - R_2) \sim \chi^2_{2(N-2)}, \tag{18}$$

so if $\kappa_1 = \kappa_2$ in equation (13) one arrives similarly at another approximation:

$$f_{\text{mc}} = \frac{(N-2)[(R_1 + R_2)^2 - R^2]}{2(N - R_1 - R_2)(R_1 + R_2)} \sim F_{2,2(N-2)}. \tag{19}$$

Had all these distributions only held conditionally, the results of McFadden & Lowes (1981) would not have the practical value which in fact they do have.

In Watson (1956), for the case of large $\kappa_1 = \kappa_2 = \kappa$, the analysis of dispersion identity

$$2\kappa(N - R) = 2\kappa[(N_1 - R_1) + (N_2 - R_2)] + 2\kappa[R_1 + R_2 - R],$$

and its approximate distributional analogue

$$\chi^2_{2(N-1)} = \chi^2_{2(N-2)} + \chi^2_2$$

led to the statistic

$$f = (N-2)(R_1 + R_2 - R)/(N - R_1 - R_2) \sim F_{2(N-2)}, \tag{20}$$

the approximate distribution being unconditional. In Watson (1983), the standard statistical likelihood ratio method and Wilk's theorem led to

$$u \equiv 2(\kappa_1 R_1 + \kappa_2 R_2 - \|\kappa_1 \mathbf{R}_1 + \kappa_2 \mathbf{R}_2\|) \sim \chi_2^2 \tag{21}$$

along with its companion

$$v \equiv 2(k_1 R_1 + k_2 R_2 - \|k_1 \mathbf{R}_1 + k_2 \mathbf{R}_2\|) \sim \chi_2^2 \tag{22}$$

again for unconditional distributions. Equation (22) is used in preference to what one obtains by following the likelihood ratio procedure (which is theoretically justified only for large samples) when the κ's are unknown. Finally in Debiche & Watson's (1991) study of the estimation of the angle θ between the centres of two Fishers—thus $\cos\theta = \boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2$, $\cos\hat{\theta} = \hat{\boldsymbol{\mu}}_1 \cdot \hat{\boldsymbol{\mu}}_2$—we found by yet other arguments that, if $\theta = 0$, then approximately

$$t'_{\hat{\theta}} \equiv (\hat{\theta}/\theta_{\text{crit}})^2 \sim \chi_2^2, \qquad \theta^2_{\text{crit}} = (k_1 R_1)^{-1} + (k_2 R_2)^{-1}. \tag{23}$$

In the simulations the quantity $(\hat{\theta}/\theta_{\text{crit}})^2$ is labelled $t_{\hat{\theta}}$ with a companion $t'_{\hat{\theta}}$ for completeness defined by

$$t'_{\hat{\theta}} \equiv (\hat{\theta}/\theta'_{\text{crit}})^2 \sim \chi_2^2, \qquad \theta'^2_{\text{crit}} = (\kappa_1 R_1)^{-1} + (\kappa_2 R_2)^{-1}. \tag{24}$$

It is seen that equation (23) is an unconditional variant of equation (11); it is not the same as equation (11) because we used a different measure of distance in the numerator. The definition in equation (10) can be written

$$1/\kappa' = 1/\kappa'_1 + 1/\kappa'_2, \qquad \kappa'_i \equiv \kappa_i R_i,$$

which is the analogue of the very familiar $\sigma^2 = \sigma_1^2 + \sigma_2^2$. This analogy was used in various guises from the beginning with directional data. It is used in Watson & Irving (1957) but appeared earlier in Runcorn (1957).

The discussion above gives us 10 statistics to compare in the next section for some selected values of N_1, N_2, κ_1, and κ_2. Table 1 should help the reader keep track of all the statistics studied and the two choices of conditions considered: κ's known or unknown, and κ's equal or unequal.

Of these statistics McFadden & Lowes (1981) suggested only f_{mc} and g for practical use. Of course those containing

Table 1. The various statistics classified by the conditions they were designed for: κ's assumed equal or not equal, and κ's assumed known or unknown. Also shown are their approximate distributions and references in which they were defined. The notation $F_{2,d}$ means F distributed with 2 and d degrees of freedom where $d = 2(N-2)$ and $N = N_1 + N_2$.

	κ's known			κ's unknown				
Condition	Sym.	Distr.	Eqn.	Sym.	Distr.	Eqn.	Reference	Remarks
$\kappa_1 = \kappa_2$	c_{12}	χ_2^2	(13')	$\hat{c}_{12}$	χ_2^2	(14')	McFadden & Lowes (1981)	a, b for $\kappa_1 = \kappa_2$
$\kappa_1 = \kappa_2$	-	-	-	f_{mc}	$F_{2,d}$	(19)	McFadden & Lowes (1981)	
$\kappa_1 = \kappa_2$	-	-	-	f	$F_{2,d}$	(20)	Watson (1956)	
$\kappa_1 \neq \kappa_2$	a	χ_2^2	(13)	b	χ_2^2	(14)	McFadden & Lowes (1981)	b is a with k instead of κ
$\kappa_1 \neq \kappa_2$	g	$F_{2,d}$	(16)	h	$F_{2,d}$	(17)	McFadden & Lowes (1981)	h is g with k instead of κ
$\kappa_1 \neq \kappa_2$	u	χ_2^2	(21)	v	χ_2^2	(22)	Watson (1983)	v is u with k instead of κ
$\kappa_1 \neq \kappa_2$	$t_{\hat{\theta}}'$	χ_2^2	(24)	$t_{\hat{\theta}}$	χ_2^2	(23)	Debiche & Watson (1991)	N, κ in $t_{\hat{\theta}}'$; R, k in $t_{\hat{\theta}}$

κ_1 and κ_2 were only considered for the insight they provide on the cost of not knowing the concentrations. Some of the above statistics, with their approximate distributions, can easily be generalized to compare the centres of any number of Fisher distributions. We will take the plausible line that the methods which work best for two populations will also work best for more than two. By 'working best' we will mean that:

(i) if the mean directions are the same the test statistic gives an accurate 5 per cent level (i.e. the approximation to the null distribution is good); and

(ii) if the mean directions are different, the test will detect the difference with a higher probability than any of the other nine test statistics.

Since it is currently impossible to make these comparisons theoretically, we must use simulation.

Another very important criterion for practical use would be the robustness to deviations from the assumption of a Fisher distribution. This requires a separate study—see e.g. the bootstrap papers of Fisher & Hall (1990, 1991). However it is our intuition that the statistics a & b and g & h, which all have squares of resultant lengths in them and a less evident geometrical basis are less likely to be robust than, say, u & v or f. Watson (1967) in a small study of the one-sample situation showed that f fares well.

2 SIMULATIONS

There are several ways to generate samples from a Fisher distribution; we used the method given in Fisher *et al.* (1987), who also explain Q–Q plots. In this study Q–Q plots based on 10 000 simulations and the F and χ^2 distributional approximations of the last section were made of most of the statistics mentioned there. If the Q–Q plot follows the $y = x$ line a little beyond the 5 per cent significance point (here roughly 3 for F with two numerator degrees of freedom and 6 for a χ^2 with two degrees of freedom), we judge the statistic a success since it means that we can rely on the results of tests at the 5 per cent level. Space limitations allow us to show only one of these graphs (Fig. 1), but the end results may be partially verified from the $\theta = 0$ line in

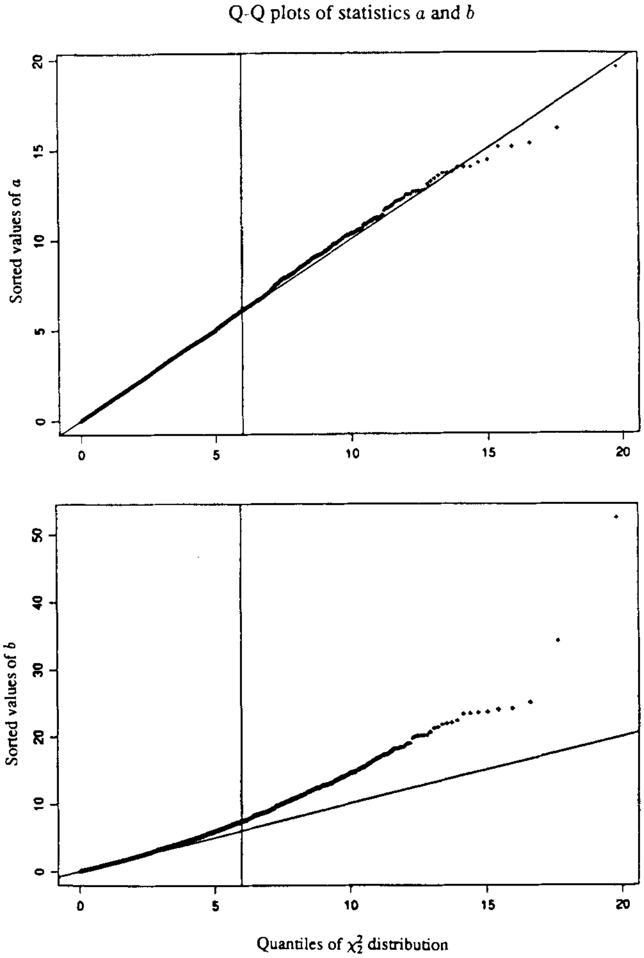

Figure 1. Q–Q plots for the statistics a and b suggested by McFadden & Lowes (1981) for a special case of 10 000 pairs of samples of size 5 from Fisher distributions with concentrations (κ's) equal to 25.

Table 2. The entries in this line of the table would be 0.05 for a perfect match to the assumed distribution.

The statistics f, f_{mc}, and c_{12} are designed only for the case $\kappa_1 = \kappa_2$ so one expects them to fail when $\kappa_1 \neq \kappa_2$ and this

Table 2. The first column is the angular separation of the true mean vectors, while the remaining columns represent the performance of the various statistics. The entries give the proportion of times in 10 000 simulations that the statistic exceeded its (approximately) predicted 5 per cent significance point. Blocks lettered (a) to (i) represent different choices of the parameters—namely, the sample sizes N_1, N_2 and the concentrations κ_1, κ_2—as shown in the block headings.

(a) $N_1 = 5, \ \kappa_1 = 25, \quad N_2 = 5, \ \kappa_2 = 25$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t'_{\hat\theta}$
0	.0466	.0458	.0489	.0489	.0733	.0496	.0460	.0458	.0450	.0555	.0743
5	.0834	.0826	.0875	.0875	.1209	.0887	.0831	.0826	.0823	.0955	.1225
10	.1745	.1724	.1993	.1993	.2363	.2013	.1736	.1724	.1719	.2164	.2393
15	.3631	.3608	.4289	.4289	.4475	.4319	.3611	.3608	.3592	.4480	.4504
20	.5967	.5925	.6796	.6796	.6785	.6825	.5945	.5925	.5900	.6988	.6821
25	.7928	.7896	.8692	.8692	.8511	.8711	.7909	.7896	.7885	.8804	.8543
30	.9151	.9134	.9594	.9594	.9446	.9601	.9141	.9134	.9121	.9640	.9460
35	.9745	.9739	.9922	.9922	.9860	.9923	.9739	.9739	.9732	.9934	.9867
40	.9944	.9940	.9987	.9987	.9963	.9987	.9941	.9940	.9937	.9989	.9968

(b) $N_1 = 5, \ \kappa_1 = 25, \quad N_2 = 10, \ \kappa_2 = 25$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t'_{\hat\theta}$
0	.0501	.0497	.0479	.0479	.0779	.0487	.0580	.0497	.0573	.0529	.0784
5	.0954	.0945	.0966	.0966	.1301	.0984	.1039	.0945	.1033	.1074	.1309
10	.2408	.2386	.2595	.2595	.2966	.2614	.2529	.2387	.2512	.2777	.2992
15	.4862	.4842	.5347	.5347	.5449	.5365	.4944	.4842	.4921	.5536	.5473
20	.7558	.7541	.8074	.8074	.7930	.8087	.7539	.7541	.7523	.8187	.7945
25	.9170	.9159	.9466	.9466	.9297	.9468	.9089	.9159	.9072	.9521	.9309
30	.9840	.9838	.9928	.9928	.9878	.9930	.9805	.9838	.9802	.9936	.9882
35	.9971	.9969	.9988	.9988	.9980	.9988	.9963	.9969	.9963	.9988	.9981
40	.9997	.9997	.9998	.9998	.9996	.9998	.9994	.9997	.9994	.9998	.9996

(c) $N_1 = 5, \ \kappa_1 = 25, \quad N_2 = 5, \ \kappa_2 = 50$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t'_{\hat\theta}$
0	.0955	.0950	.0300	.0529	.0905	.0531	.0696	.0528	.0688	.0570	.0908
5	.1668	.1662	.0655	.1082	.1528	.1089	.1233	.1034	.1228	.1185	.1536
10	.3891	.3876	.2338	.3162	.3506	.3167	.3062	.2843	.3056	.3330	.3527
15	.6831	.6820	.5251	.6179	.6310	.6198	.5802	.5686	.5786	.6351	.6331
20	.9002	.8999	.8256	.8787	.8610	.8796	.8273	.8321	.8257	.8884	.8621
25	.9779	.9774	.9596	.9763	.9610	.9765	.9484	.9586	.9479	.9792	.9620
30	.9972	.9972	.9958	.9981	.9941	.9981	.9903	.9940	.9901	.9982	.9944
35	.9997	.9997	.9999	.9999	.9994	.9999	.9989	.9995	.9989	.9999	.9995
40	1.0000	1.0000	1.0000	1.0000	.9999	1.0000	.9999	.9999	.9999	1.0000	.9999

(d) $N_1 = 5, \ \kappa_1 = 25, \quad N_2 = 5, \ \kappa_2 = 100$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t'_{\hat\theta}$
0	.1432	.1426	.0185	.0502	.0924	.0503	.0739	.0485	.0738	.0541	.0932
5	.2476	.2467	.0544	.1197	.1697	.1198	.1435	.1139	.1430	.1294	.1709
10	.5167	.5155	.2135	.3426	.3886	.3431	.3457	.3122	.3449	.3594	.3902
15	.7903	.7897	.5200	.6648	.6659	.6655	.6186	.6136	.6183	.6829	.6679
20	.9498	.9494	.8305	.9084	.8868	.9085	.8585	.8687	.8581	.9149	.8878
25	.9936	.9936	.9696	.9885	.9772	.9887	.9656	.9753	.9653	.9899	.9779
30	.9998	.9998	.9966	.9990	.9965	.9990	.9944	.9972	.9943	.9991	.9967
35	1.0000	1.0000	.9999	1.0000	.9994	1.0000	.9991	1.0000	.9991	1.0000	.9995
40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(e) $N_1 = 10, \ \kappa_1 = 25, \quad N_2 = 10, \ \kappa_2 = 50$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t'_{\hat\theta}$
0	.0511	.0505	.0504	.0504	.0623	.0509	.0509	.0505	.0503	.0573	.0624
5	.1173	.1169	.1233	.1233	.1379	.1236	.1172	.1169	.1165	.1359	.1389
10	.3545	.3532	.3785	.3785	.3918	.3798	.3538	.3532	.3523	.4001	.3952

Table 2. (*continued*)

(e) $N_1 = 10, \kappa_1 = 25, \quad N_2 = 10, \kappa_2 = 50$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t_{\hat\theta}'$
15	.6990	.6982	.7380	.7380	.7352	.7398	.6985	.6982	.6978	.7546	.7382
20	.9123	.9115	.9361	.9361	.9304	.9365	.9119	.9115	.9111	.9418	.9313
25	.9894	.9892	.9944	.9944	.9917	.9945	.9892	.9892	.9887	.9947	.9917
30	.9990	.9990	.9994	.9994	.9994	.9995	.9990	.9990	.9990	.9996	.9994
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(f) $N_1 = 10, \kappa_1 = 25, \quad N_2 = 10, \kappa_2 = 100$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t_{\hat\theta}'$
0	.0553	.0548	.0191	.0498	.0667	.0502	.0546	.0519	.0544	.0557	.0671
5	.1456	.1449	.0762	.1516	.1690	.1522	.1436	.1431	.1433	.1628	.1702
10	.4642	.4632	.3382	.4927	.5019	.4935	.4620	.4609	.4612	.5103	.5025
15	.8257	.8254	.7527	.8554	.8487	.8561	.8234	.8239	.8228	.8649	.8496
20	.9734	.9732	.9593	.9841	.9784	.9842	.9730	.9730	.9729	.9859	.9787
25	.9987	.9985	.9978	.9996	.9992	.9996	.9984	.9989	.9984	.9996	.9992
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(g) $N_1 = 10, \kappa_1 = 25, \quad N_2 = 10, \kappa_2 = 100$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t_{\hat\theta}'$
0	.0570	.0568	.0083	.0473	.0688	.0474	.0555	.0486	.0553	.0522	.0692
5	.1703	.1697	.0508	.1686	.1952	.1688	.1680	.1624	.1676	.1809	.1963
10	.5451	.5446	.3235	.5812	.5822	.5815	.5405	.5375	.5401	.5971	.5836
15	.8845	.8840	.7528	.9138	.9014	.9139	.8819	.8826	.8813	.9199	.9023
20	.9890	.9889	.9721	.9946	.9907	.9946	.9884	.9891	.9883	.9953	.9907
25	.9997	.9997	.9988	.9998	.9999	.9998	.9997	.9998	.9997	.9999	.9999
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(h) $N_1 = 10, \kappa_1 = 50, \quad N_2 = 10, \kappa_2 = 50$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t_{\hat\theta}'$
0	.0498	.0497	.0496	.0496	.0614	.0497	.0497	.0497	.0496	.0528	.0617
5	.1965	.1965	.2080	.2080	.2252	.2086	.1964	.1965	.1961	.2163	.2267
10	.6456	.6452	.6816	.6816	.6800	.6825	.6453	.6452	.6445	.6907	.6813
15	.9481	.9481	.9626	.9626	.9575	.9627	.9481	.9481	.9480	.9643	.9578
20	.9983	.9982	.9993	.9993	.9989	.9993	.9983	.9982	.9982	.9993	.9989
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(i) $N_1 = 10, \kappa_1 = 50, \quad N_2 = 10, \kappa_2 = 100$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t_{\hat\theta}'$
0	.0523	.0522	.0191	.0512	.0653	.0513	.0523	.0500	.0522	.0538	.0654
5	.2517	.2511	.1548	.2713	.2876	.2717	.2504	.2498	.2500	.2786	.2884
10	.7798	.7795	.6935	.8154	.8100	.8157	.7788	.7802	.7786	.8209	.8102
15	.9845	.9845	.9778	.9907	.9881	.9907	.9844	.9851	.9844	.9916	.9882
20	.9999	.9999	.9999	1.0000	.9999	1.0000	.9999	.9999	.9999	1.0000	.9999
25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(j) $N_1 = 10, \kappa_1 = 100, \quad N_2 = 10, \kappa_2 = 100$

θ	f	f_{mc}	c_{12}	a	b	u	v	g	h	$t_{\hat\theta}$	$t_{\hat\theta}'$
0	.0493	.0492	.0483	.0483	.0629	.0483	.0493	.0492	.0492	.0499	.0629
5	.3765	.3763	.4021	.4022	.4158	.4023	.3763	.3763	.3758	.4082	.4162
10	.9229	.9226	.9416	.9416	.9360	.9417	.9228	.9226	.9224	.9443	.9360
15	.9997	.9997	1.0000	1.0000	.9998	1.0000	.9997	.9997	.9997	1.0000	.9998
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

was observed. When $\kappa_1 = \kappa_2$, not only did they pass but only b did badly for the following combinations:

N_1	N_2	κ_1	κ_2
5	5	25	25
5	10	25	25
5	10	25	50
10	10	50	50
10	10	100	100

One can verify this partly from the $\theta = 0$ line in Table 1. Of the practical statistics for the case $\kappa_1 \neq \kappa_2$, b, v, h and $t_{\hat{\theta}}$, it seems that the nominal points of v and h are the most accurate.

Table 2 gives the power functions of the tests based upon the various statistics and uses significance points from their approximate distributions. The suggested tests based on v and h give nearly identical results! Comparing the columns for u, v, g and h, one can see that u does much better than g but v only trivially better than h. It seems that a bigger price is paid for not knowing the κ's in the (u, v) pair. The power functions of g and h are virtually the same. We don't know why this is so; however, it does mean that, in the rare event that the κ's are known, u should be used. If one knows that $\kappa_1 = \kappa_2$, one should never use c_{12}, and further f is only trivially better than f_{mc}. But their power functions do not dominate those of v and h so that there is a real argument for using one of v or h. We were however surprized at the robustness of both f and f_{mc} to differing κ's (see Table 2). The most surprizing result was the poor performance of b and $t_{\hat{\theta}}$.

We also plotted the differences of the power functions in Figs 2–4. For known κ_1 and κ_2, the difference D was

$D =$ (power of statistic) $-$ (power of u).

For unknown κ_1 and κ_2, the difference D was

$D =$ (power of statistic) $-$ (power of v).

Fig. 2 shows one example when u was the standard; one can see that here a is essentially the same as u, g is substantially worse and $t'_{\hat{\theta}}$ would be a little better if one could get its significance point more accurately.

Figures 3 and 4 show that the useful statistics always fall in three pairs: $t_{\hat{\theta}}$ & b, f & f_{mc}, and v & h. They show further than f, f_{mc}, h and v behave very well if the sample sizes are equal, but not otherwise.

3 DISCUSSION

For comparing the mean directions of two Fisher distributions, our simulations suggest that either the statistic h of equation (17) or v of (22) may be used in the practical case where one is not certain that the κ's are the same. If a user is concerned about this conclusion for smaller κ's or N's then we suggest that a special simulation be run—with today's computers this is easy to do. Though the power functions of these tests were found, they could not be used

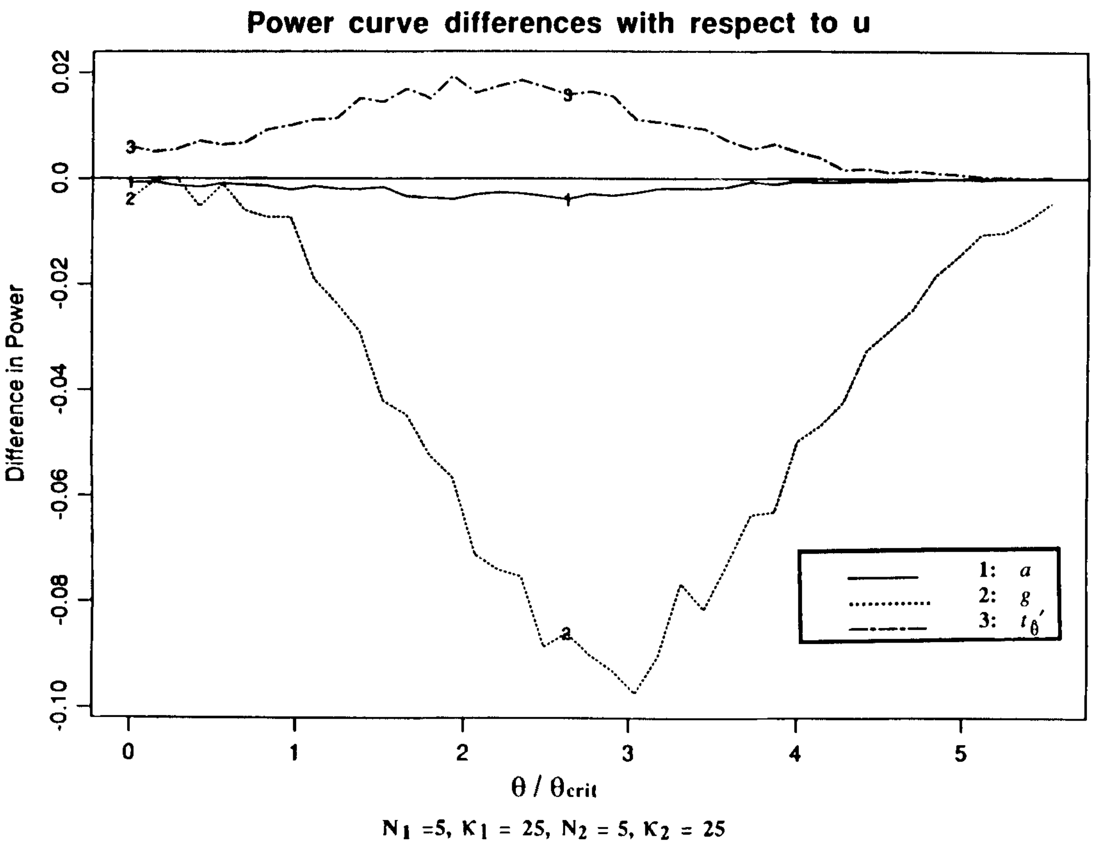

Figure 2. Powers of statistics a, g, and $t'_{\hat{\theta}}$ minus power of u plotted against θ/θ_{crit} [see equation (24)]. 10 000 pairs of samples of size 5 were drawn from Fisher distributions with κ's equal to 25.

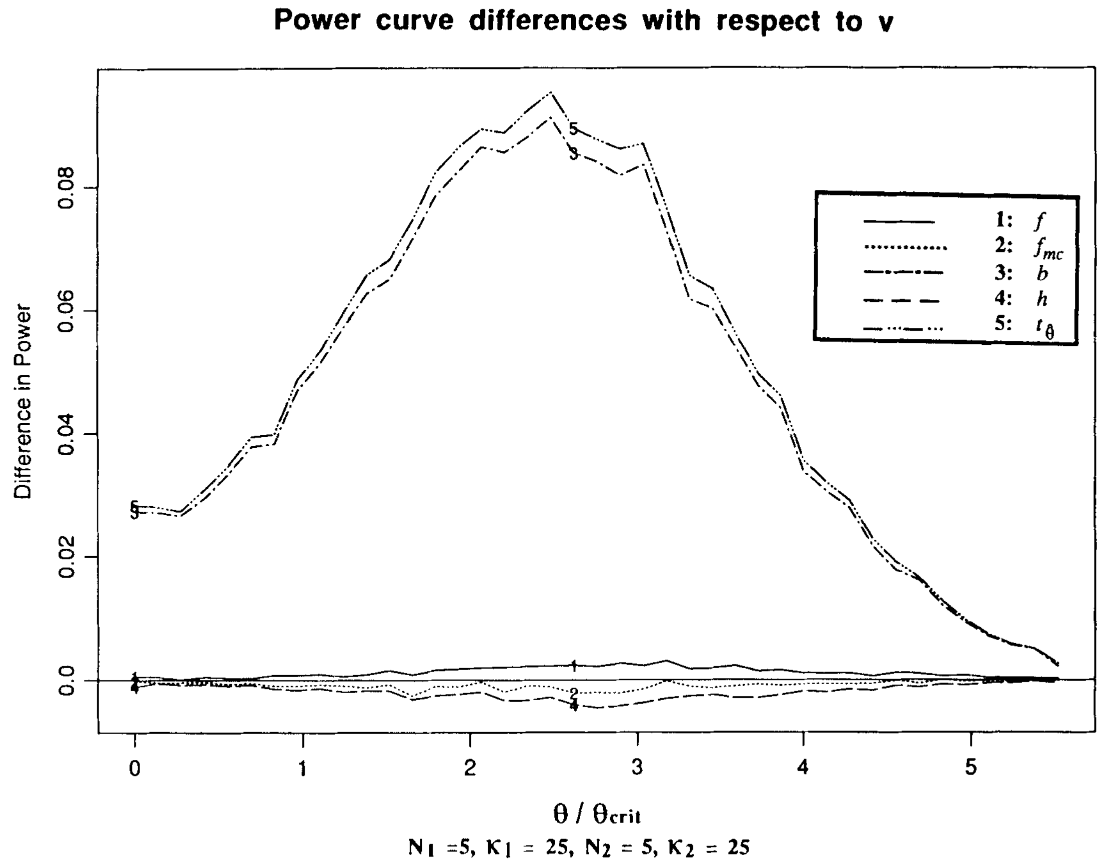

Figure 3. Powers of statistics f, f_{mc}, b, h and $t_{\hat{\theta}}$ minus power of v plotted against θ/θ_{crit} [see equation (24)]. The 10 000 simulations used samples each of size 5 from Fisher distributions with κ's equal to 25.

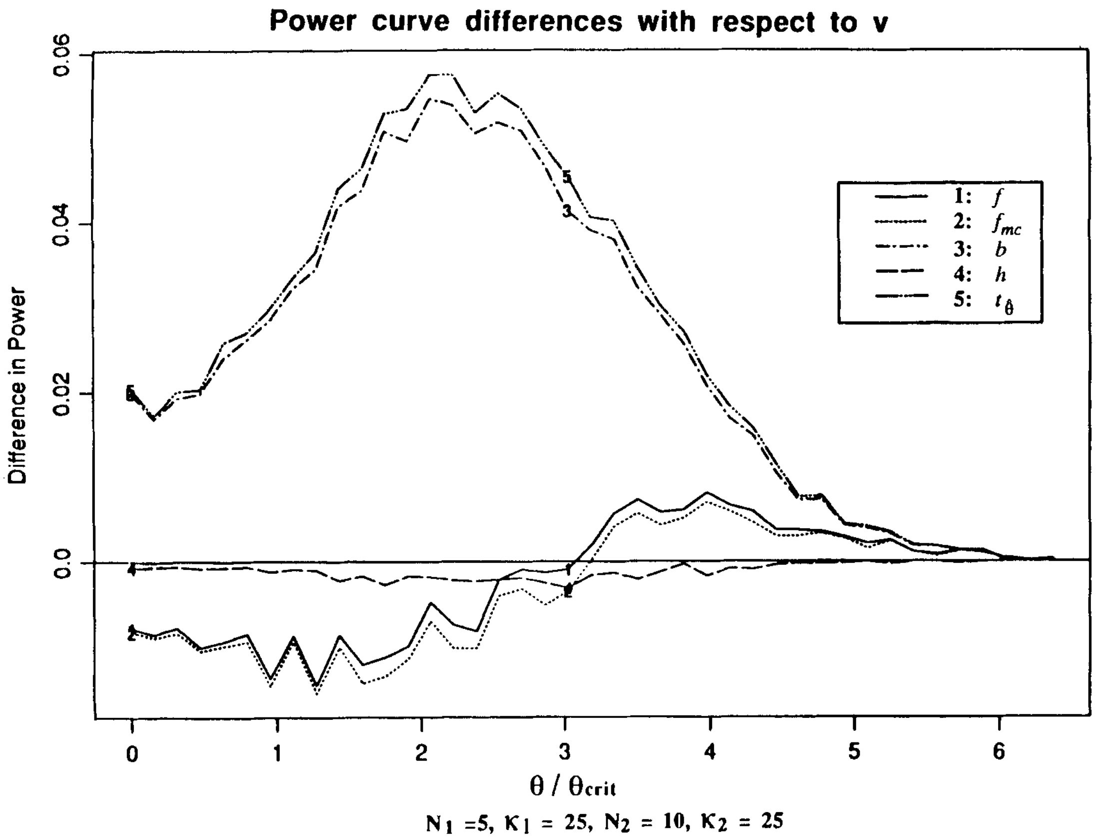

Figure 4. Powers of statistics f, f_{mc}, b, h and $t_{\hat{\theta}}$ minus power of v plotted against θ/θ_{crit} [see equation (24)]. The 10 000 simulations used samples of sizes 5 and 10 from Fisher distributions with κ's equal to 25.

to estimate required sample sizes; however, they can give one a rough idea which should be helpful. We suggest that simulation be used to check projected choices of sample sizes (preferably equal) before the data are collected. This is an easy task since only one statistic need be studied.

The statistics u and v generalize naturally to more than two distributions. If we may extrapolate from the simulations of Section 2 to the m sample problem, it seems that one should recommend the use of

$$U = \sum \kappa_i R_i - \left\| \boldsymbol{\sum} \kappa_i \mathbf{R}_i \right\| \sim \chi^2_{2(m-1)}, \qquad \kappa_i\text{'s known}, \tag{25}$$

$$V = \sum k_i R_i - \left\| \boldsymbol{\sum} k_i \mathbf{R}_i \right\| \sim \chi^2_{2(m-1)}, \qquad \kappa_i\text{'s unknown}. \tag{26}$$

Here $i = 1, \ldots, m$ and the boldface $\boldsymbol{\sum}$ denotes vector summation. These statistics have the extra support that they are more likely to be robust, partly because they have a simple geometric interpretation, and that their large sample theory has been studied (Watson 1983). Of course these informed guesses could be checked by simulation in any particular case.

4 CONCLUSIONS

For comparing the mean directions of two Fisher distributions, our simulations suggest that either the statistic h of equation (17) or v of (22) may be used in the usual case where the concentrations are unknown. We suggest a natural generalization of the statistic v for comparing the mean directions of more than two Fisher distributions. Furthermore our work shows that every effort should be made to have comparably sized samples.

ACKNOWLEDGMENTS

The work of Watson was partially supported by the NSF Grant 8803207. Debiche was supported by a National Science and Engineering Research Council of Canada post-graduate fellowship.

REFERENCES

Debiche, M. G. & Watson, G. S., 1991. Estimating angles between directions, *J. geophys. Res.*, in press.

Fisher, R. A., 1953. Dispersion on a sphere, *Proc. R. Soc. London.*, A, **217,** 295–305.

Fisher, N. & Hall, P., 1990. New statistical methods for directional data—1. Bootstrap comparison of mean directions and the fold test in palaeomagnetism, *Geophys. J. Int.*, **101,** 305–313.

Fisher, N. & Hall, P., 1991. A general statistical test for the effect of folding, *Geophys. J. Int.*, **105,** 419–427.

Fisher, N., Lewis, T. & Embleton, B., 1987. *Statistical Analysis of Spherical Data*, Cambridge University Press, Cambridge, UK.

McFadden, P. L. & Lowes, F. J., 1981. The discrimination of mean directions drawn from Fisher distributions, *Geophys. J. R. astr. Soc.*, **67,** 19–33.

Runcorn, S. K., 1957. The sampling of rocks for paleomagnetic comparisons between the continents, *Adv. Phys.*, **6,** 169–176.

Watson, G. S., 1956. Analysis of dispersion on a sphere, *Mon. Not. R. astr. Soc., Geophys. Suppl.*, **7,** 153–159.

Watson, G. S., 1967. Some problems in the statistics of directions, *Bull. Int. Stat. Inst.*, **42,** 374–385.

Watson, G. S., 1983. *Statistics on Spheres*, University of Arkansas Lecture Notes in the Mathematical Sciences, Wiley, New York.

Watson, G. S. & Irving, E., 1957. Statistical methods in rock magnetism, *Mon. Not. R. astr. Soc., Geophys. Suppl.*, **7,** 289–300.