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The discrimination of mean directions drawn from Fisher distributions

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Summary. An acceptably accurate approximation for the sampling distribution of the angle between two sample mean directions, conditional on the observed lengths of the vector resultants, is derived for samples drawn from Fisher populations sharing a common true mean direction. From this a test is given for the null hypothesis that two populations (with a common precision parameter) share a common true mean direction. This test is then compared with the unconditional test derived by Watson.

The conditional test is then extended to an approximate test for the case where the two populations do not share a common precision parameter.

The conditional test for populations with a common precision parameter is then extended to the case where it is desired to test simultaneously whether several samples could have been drawn from populations sharing a common true mean direction.

The pooled, unbiased estimate for the inverse of the precision parameter is determined. From this a test for homogeneity of the precision parameter is derived for the case of several samples having unequal sample sizes.

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1 Introduction

Fisher's (1953) distribution of vectors on a unit sphere is given by

$$P(A)\,dA = \frac{\kappa}{4\pi \sinh \kappa} \exp(\kappa \cos \theta)\, dA. \tag{1}$$

Here κ is the precision parameter, θ the polar angle between the true mean direction and an observed direction and dA is the element of area at (θ, ϕ), where ϕ is the uniformly distributed azimuthal angle. The marginal distribution of θ is thus

$$P(\theta)\,d\theta = \frac{\kappa}{2 \sinh \kappa} \exp(\kappa \cos \theta) \sin \theta \, d\theta. \tag{2}$$

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Fisher (1953) has shown that the maximum likelihood estimate (from a random sample of N unit vectors) for the true mean direction is given by the direction of the vector resultant, length R, of the individual unit vectors. Further, he has shown that if α is the angle between this sample mean direction and the true mean direction, then the distribution of α conditional on given R, $P(\alpha \mid R)$, is

$$P(\alpha \mid R)\, d\alpha = \frac{R\kappa}{2 \sinh (R\kappa)} \exp(R\kappa \cos \alpha) \sin \alpha d\alpha. \tag{3}$$

It is of great value, particularly in palaeomagnetic studies, to be able to test whether two such observed mean direction could have been obtained by random sampling from populations having the same mean directions. Several exact tests have been suggested, but it would be useful to have a simple test valid for reasonably large κ.

Such a test has been derived by Watson (1956) for the case where populations have the same precision parameter, but unfortunately his test does not lead to a test for populations which do not share a common precision parameter. It is the intention here to develop a conditional test in a manner in which it is applicable not only to populations sharing a common precision parameter but also (as an approximate test) to populations without a common precision parameter.

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2 Sampling distribution of the angle between two mean directions

In order to derive the required test it is first necessary to derive the sampling distribution of the angle between two mean directions, conditional on the observed lengths of the vector resultants of the two samples and conditional on the two populations sharing a common true mean direction. The strict derivation is complicated and difficult to perform. The following analysis is a simple method of elucidating an acceptably accurate approximation to the required distribution; it uses the standard result that, for large κ, the Fisher distribution on the unit sphere is well approximated by a bivariate normal distribution on the tangent plane.

The distribution of α given by Fisher (1953) is in fact the marginal distributional conditional on given R, i.e. independent of the azimuthal angle ϕ. The joint distribution, conditional on given R, is therefore

$$P(\alpha, \phi \mid R)\, dA = \frac{R\kappa}{4\pi \sinh (R\kappa)} \exp(R\kappa \cos \alpha)\, dA. \tag{4}$$

For cases of practical interest the term $(R\kappa)$ is large enough that α is effectively restricted to small values and so

$$\cos \alpha \cong 1 - \tfrac{1}{2}\alpha^2 \text{ and } 2 \sinh (R\kappa) \cong \exp(R\kappa). \tag{5}$$

Hence the joint distribution is given, to a good approximation, by

$$P(\alpha, \phi \mid R)\, dA \cong \frac{R\kappa}{2\pi} \exp(-\tfrac{1}{2}R\kappa\alpha^2)\, dA, \tag{6}$$

which is the bivariate normal distribution of two independent variables sharing a common variance $(1/R\kappa)$. This may be considered as a projection of the Fisher distribution on to a plane tangential to the unit sphere, the point of contact being the true mean direction. Over the region of interest the distance between two points on this plane is equal to the angle (in radians) subtended by these two points at the centre of the unit sphere. Defining two new variables (x and y) in this plane by the relations

$$\alpha^2 = x^2 + y^2$$

and (7)

$$\phi = \arctan(y/x)$$

the variables x and y are orthogonal, independent and have marginal distributions which are normal, mean zero, variance $(1/R\kappa)$, i.e.

$$g(x)\,dx = \left(\frac{R\kappa}{2\pi}\right)^{1/2} \exp(-\tfrac{1}{2}R\kappa x^2)\,dx$$

and (8)

$$h(y)\,dy = \left(\frac{R\kappa}{2\pi}\right)^{1/2} \exp(-\tfrac{1}{2}R\kappa y^2)\,dy.$$

The true mean direction is now located at the point (0, 0) in the plane, and the angle between two directions (α_1, ϕ_1) and (α_2, ϕ_2) is simply equal to the distance between the Cartesian coordinates (x_1, y_1) and (x_2, y_2).

It is convenient at this stage to introduce a shorthand notation whereby equations (8) may be rewritten as

$$x \sim N(0, 1/R\kappa); \quad y \sim N(0, 1/R\kappa). \tag{9}$$

Here the symbol '$\sim$' is to be read as 'is distributed as' and $N(a, b)$ indicates a normal distribution, mean a, variance b.

Consider now two Fisher populations having the same true mean direction but different precision parameters, κ_1 and κ_2. A sample of size N_1 drawn from the first population will give a mean direction (α_1, ϕ_1) or equivalently (x_1, y_1) and, conditional on the observed value R_1,

$$x_1 \sim N(0, 1/R_1\kappa_1); \quad y_1 \sim N(0, 1/R_1\kappa_1). \tag{10}$$

Similarly, a sample of size N_2 drawn from the second population will give a mean direction (α_2, ϕ_2) or equivalently (x_2, y_2) and, conditional on the observed value R_2,

$$x_2 \sim N(0, 1/R_2\kappa_2); \quad y_2 \sim N(0, 1/R_2\kappa_2). \tag{11}$$

Hence, using the reproductive property of the normal distribution, conditional on the observed values R_1 and R_2,

$$(x_2 - x_1) \sim N(0, \sigma^2); \quad (y_2 - y_1) \sim N(0, \sigma^2) \tag{12}$$

where

$$\sigma^2 = \frac{1}{R_1\kappa_1} + \frac{1}{R_2\kappa_2}. \tag{13}$$

It follows immediately that

$$\frac{(x_2 - x_1)^2}{\sigma^2} \sim \chi_1^2; \quad \frac{(y_2 - y_1)^2}{\sigma^2} \sim \chi_1^2. \tag{14}$$

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Further, since the variables $(x_2 - x_1)$ and $(y_2 - y_1)$ are independent,

$$\frac{(x_2-x_1)^2 + (y_2-y_1)^2}{\sigma^2} \sim \chi_2^2. \tag{15}$$

The angle, γ, between the two mean directions is given by

$$\gamma^2 = (x_2-x_1)^2 + (y_2-y_1)^2 \tag{16}$$

and so

$$\frac{\gamma^2}{\sigma^2} = \kappa'\gamma^2 \sim \chi_2^2 \tag{17}$$

where

$$\kappa' = \frac{1}{\sigma^2} = \frac{\kappa_1\kappa_2 R_1 R_2}{\kappa_1 R_1 + \kappa_2 R_2}. \tag{18}$$

Equations (17) and (18) give the approximate distribution of γ conditional on the observed values of R_1 and R_2. However, it is of more value to rewrite this in terms of the distribution of R (the length of the vector resultant of the $(N_1 + N_2)$ individual unit vectors) conditional on the observed values of R_1 and R_2. Using the approximation $(\cos\gamma = 1 - \frac{1}{2}\gamma^2)$, the value of γ^2 is given by

$$\gamma^2 = \frac{1}{R_1 R_2}((R_1 + R_2)^2 - R^2). \tag{19}$$

Hence on substituting into (17)

$$\frac{\kappa_1\kappa_2((R_1+R_2)^2 - R^2)}{R_1\kappa_1 + R_2\kappa_2} \sim \chi_2^2. \tag{20}$$

3 Comparison of two mean directions

Watson (1956) has shown that

$$2\kappa(N_i - R_i) \sim \chi^2_{2(N_i-1)} \tag{21}$$

(see McFadden (1980) for a discussion of the range of validity of this approximation) and since the two samples at present under consideration are independent it follows that

$$2\kappa_1(N_1 - R_1) + 2\kappa_2(N_2 - R_2) \sim \chi^2_{2(N-2)}, \tag{22}$$

where $N = N_1 + N_2$. If the two true mean directions are the same it follows from (20) and (22) that

$$\frac{(N-2)\kappa_1\kappa_2((R_1+R_2)^2 - R^2)}{(2\kappa_1(N_1-R_1) + 2\kappa_2(N_2-R_2))(R_1\kappa_1 + R_2\kappa_2)} = f \sim F[2, 2(N-2)], \tag{23}$$

where $F[m, n]$ is the F distribution with m and n degrees of freedom respectively. This then is the general form of the test and if the observed value of f exceeds the critical value of the F distribution at the required level of significance then the null hypothesis that the two true

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mean direction are the same may be rejected. In general, however, the parameters κ_1 and κ_2 are unknown and so it is still necessary to eliminate these parameters from the test.

3.1 CASE WHEN $\kappa_1 = \kappa_2$

Initially it is necessary that there be some causal reason for expecting $\kappa_1 = \kappa_2$. Given that such causal reason exists a statistical test of the null hypothesis that $\kappa_1 = \kappa_2$ should be performed. Defining the statistic k_i as

$$k_i = \frac{N_i - 1}{N_i - R_i} \tag{24}$$

the statistic $(1/k_i)$ is then an unbiased estimator for $(1/\kappa_i)$ (McFadden 1980) and the ratio (k_1/k_2) may be used to test for disparity between κ_1 and κ_2 using the statistic

$$\frac{k_1}{k_2} = \frac{(N_1 - 1)}{(N_1 - R_1)} \frac{(N_2 - R_2)}{(N_2 - 1)} = h \sim F[2(N_2 - 1), 2(N_1 - 1)]. \tag{24a}$$

If the hypothesis that $\kappa_1 = \kappa_2$ cannot be rejected at the required level of significance then it may be accepted that $\kappa_1 = \kappa_2 = \kappa$. With this condition equation (23) reduces to

$$(N-2) \frac{(R_1 + R_2 - R^2/(R_1 + R_2))}{2(N - R_1 - R_2)} = f \sim F[2, 2(N-2)]. \tag{25}$$

The statistic f is simply the ratio of γ^2 (where γ is the angle between the two mean directions) to the estimate of its expected mean square value (i.e. $2/\kappa'$), which is obtained using the unbiased, pooled, estimate for $(1/\kappa)$ of (64) in (18).

If the observed value of f exceeds the critical value of the F distribution at the required level of significance then the null hypothesis that the two true mean directions are the same may be rejected.

In fact the density, $F(f;\, 2,\, 2(N-2))$, of the distribution $F[2,\, 2(N-2)]$ is particularly simple and is given by

$$F(f; 2, 2(N-2)) = \left(1 + \frac{f}{N-2}\right)^{-(N-1)}. \tag{26}$$

Hence the value, f_p, which f will exceed with probability p is given by solving

$$p = \int_{f_p}^{\infty} \left(1 + \frac{f}{N-2}\right)^{-(N-1)} df, \tag{27}$$

so that

$$f_p = (N-2) \left[\left(\frac{1}{p}\right)^{1/(N-2)} - 1\right]. \tag{28}$$

Thus, from (25) and (28), the null hypothesis that the two true mean directions are the same may be rejected at the level of significance p if

$$\frac{(R_1 + R_2 - R^2/(R_1 + R_2))}{2(N - R_1 - R_2)} > \left(\frac{1}{p}\right)^{1/(N-2)} - 1. \tag{29}$$

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It should be noted that this test is independent of κ (i.e. it takes into account all values of κ) and is therefore directly analogous to the analysis of variance in normally distributed statistics.

3.1.1 Comparison with Watson's analysis for the case $\kappa_1 = \kappa_2$

Watson (1956, equation 2.31) noted that

$$2\kappa(N-R) = 2\kappa(N_1-R_1) + 2\kappa(N_2-R_2) + 2\kappa(R_1+R_2-R) \tag{30}$$

and using the distributions given by (21) and (22) showed that

$$2\kappa(R_1+R_2-R) \sim \chi_2^2. \tag{31}$$

The expression given here is

$$\left(1+\frac{R}{R_1+R_2}\right)\kappa\,(R_1+R_2) \sim \chi_2^2. \tag{20a}$$

The difference is that Watson's analysis is for the unconditional distribution whereas the distribution given here is that conditional on R_1 and R_2. This is easily seen since in Watson's (1956) analysis R can attain the value N, whereas, given the observed values R_1 and R_2, it is not possible for R to exceed (R_1+R_2).

Since R cannot exceed (R_1+R_2), Watson's statistic (his (2.32)) will be slightly larger than the statistic given here. However, in the region of interest the numerical difference is very small and it matters little which test is used. If the test given here is used it must be interpreted as being conditional upon the observed values of R_1 and R_2 and for that reason it is preferred by the present authors. In addition, the conditional test given here leads directly to an approximate test for the case where $\kappa_1 \neq \kappa_2$, as shown in the next section.

3.2 CASE WHEN $\kappa_1 \neq \kappa_2$

Occasionally it is necessary to test the hypothesis that the true mean directions of two populations are not different even though their precision parameters are different.

Given that

$$\kappa_2 = r\,\kappa_1 \tag{32}$$

then, from (23) and (28), the null hypothesis of a common true mean direction may be rejected at the level of significance p if

$$\frac{r((R_1+R_2)^2-R^2)}{2((N_1-R_1)+r(N_2-R_2))\,(R_1+rR_2)} > \left(\frac{1}{p}\right)^{1/(N-2)} - 1. \tag{33}$$

If causal reason exists for expecting a certain value of r then the hypothesis that $\kappa_2 = r\,\kappa_1$ may be tested using the statistic.

$$\frac{rk_1}{k_2} \sim F[2(N_2-1), 2(N_1-1)] \tag{34}$$

(McFadden 1980, equation (28)). Evidently $r = 1$ is merely the case considered in Section 3.1. However, if there is no causal reason for expecting a certain value of r, or if the hypothesis of an expected value of r has been rejected statistically, then r constitutes an unknown nuisance parameter.

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At the present time no exact test exists for this situation. However, an approximate test is readily available by using an estimate, $r_e = k_2/k_1$, for r in equation (33). This estimate is discussed in Appendix A. The usefulness of this approximate test arises from the fact that, as suggested by the example of Appendix B, the angle between the two observed mean directions at which the null hypothesis of a common true mean direction would be rejected is relatively insensitive to the value of r. However, since the test is approximate, a decision should be deferred in marginal cases.

4 Extension to several mean directions

The extension to several mean directions is made only for the case where the populations share a common precision parameter. If the precision parameters differ the test must be restricted to two mean direction. The test for homogeneity of precision is presented in Section 5.

Given three observed mean directions, from equation (17) and (18)

$$\begin{aligned} &\kappa_{12}(\gamma_{12})^2 \sim \chi_2^2 \\ &\kappa_{13}(\gamma_{13})^2 \sim \chi_2^2 \\ &\kappa_{23}(\gamma_{23})^2 \sim \chi_2^2 \end{aligned} \tag{35}$$

where

$$\kappa_{ij} = \kappa \frac{R_i R_j}{R_i + R_j}$$

and γ_{ij} is the angle between the ith mean direction and the jth mean direction. Any two of the chi-square distributed statistics in equation (35) are independent and so

$$\begin{aligned} &\kappa_{12}(\gamma_{12})^2 + \kappa_{13}(\gamma_{13})^2 = A \sim \chi_4^2 \\ &\kappa_{12}(\gamma_{12})^2 + \kappa_{23}(\gamma_{23})^2 = B \sim \chi_4^2 \\ &\kappa_{13}(\gamma_{13})^2 + \kappa_{23}(\gamma_{23})^2 = C \sim \chi_4^2. \end{aligned} \tag{36}$$

Now A, B and C are equivalent statistics, given R_1, R_2, R_3 *and* their observed orientations, the only difference being the arbitrary choice of numbering the mean directions. Thus

$$\frac{1}{3}(A+B+C) \sim \chi_4^2 \tag{37}$$

which, given that κR_1, κR_2 and κR_3 are all large, reduces to

$$\kappa\left(R_1+R_2+R_3-\frac{R^2}{R_1+R_2+R_3}\right) = D \sim \chi_4^2. \tag{38}$$

The statistic D is independent of the arbitrary numbering choice and its distribution gives the required distribution of R (the length of the overall vector resultant) conditional on R_1, R_2 and R_3. Extension of this argument shows that for m mean directions the required distribution is

$$\kappa\left({}_i\Sigma R_i - \frac{R^2}{{}_i\Sigma R_i}\right) \sim \chi^2_{2(m-1)}, \tag{39}$$

the summation running over i from 1 to m.

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Extending relation (22) it follows that

$$2\kappa(N - {}_i\Sigma R_i) \sim \chi^2_{2(N-m)}, \tag{40}$$

where

$$N = \sum_i N_i.$$

Hence from (39) and (40), under the null hypothesis that all the samples are drawn from populations having the same true mean direction (and, as before, subject to the hypothesis that the populations have a common precision parameter), the distribution independent of κ is

$$\frac{(N-m)}{(m-1)} \cdot \frac{{}_i\Sigma R_i - R^2/\Sigma_i R_i}{2(N - {}_i\Sigma R_i)} = g \sim F[2(m-1), 2(N-m)]; \tag{41}$$

this is the extension of (25) to the case of many samples. The statistic g is effectively the mean of the ratios of the individual γ^2_{ij} to the estimates of their expected mean square values. If the observed value of g exceeds the critical value of the F distribution at the required level of significance then the null hypothesis of a common true mean direction may be rejected.

Watson (1956, equation 3.4) and Watson & Williams (1956, equation 22) extended Watson's unconditional test to the multi-sample case. This is quoted by Stephens (1969, section 2.2 and, after noting that the exact distribution of the R_1 conditional on given R leads to an intractable joint density function, in section 3.3) and by Mardia (1972, equation 9.5.3). As for the two sample case the present authors prefer the conditional test for the multi-sample case; Mardia also includes an (empirical) correction factor for small κ, but this is to improve the approximation of our (21) and does not affect the conditionality.

5 Testing for homogeneity of precision

If the sample sizes are all the same the test is extremely simple. The largest observed value of $(1/k_i)$ (i.e. estimate of $(1/\kappa)$, see equation 24) is tested against the smallest observed value of $(1/k_i)$ (see equation 24a). If these two values could have been obtained by random sampling from populations having a common precision then so could all the intermediate values of $(1/k_i)$. However, if the result is marginal, or if the sample sizes vary then each of the observed $(1/k_i)$ must be considered and the required test is derived below.

5.1 GENERALIZED LIKELIHOOD RATIO

Suppose there are m samples $(m > 2)$ and that for the ith sample, of size N_i, the observed directions have polar angles θ_{ij} $(i = 1, \ldots, m; j = 1, \ldots, N_i)$ with respect to the true mean direction of that population, which has direction cosines (p_i, q_i, r_i). (The true mean directions are unknown, but it turns out that only estimated values of the θ_{ij} are needed.) Further suppose that all the samples are independent and are drawn from Fisher distributions with precisions κ_i. The null hypothesis H_0 is given by

$$H_0: \kappa_1 = \kappa_2 = \ldots = \kappa_m = \kappa$$

and the alternative hypothesis H_1 is that these precisions are not all equal.

From (1), the likelihood function under H_0 is

$$L_0 = \frac{\kappa^N}{(4\pi)^N (\sinh\kappa)^N} \exp\left(\sum_i \sum_j \kappa \cos\theta_{ij}\right) \tag{42}$$

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where

$$N = \sum_i N_i.$$

Under H_1 the likelihood function is

$$L_1 = \frac{\kappa_1^{N_1}\,\kappa_2^{N_2}\ldots\kappa_m^{N_m}}{(4\pi)^N(\sinh\kappa_1)^{N_1}\ldots(\sinh\kappa_m)^{N_m}}\exp\left(\sum_i\sum_j \kappa_i\cos\theta_{ij}\right). \tag{43}$$

The likelihood ratio (i.e. the ratio L_0/L_1 when the maximum likelihood estimators are used) is then

$$\frac{L_0\,(\max)}{L_1\,(\max)} = \frac{\hat{k}^N}{\hat{k}_1^{N_1}\,\hat{k}_2^{N_2}\ldots\hat{k}_m^{N_m}}\cdot\frac{(\sinh\hat{k}_1)^{N_1}\ldots;(\sinh\hat{k}_m)^{N_m}}{(\sinh\hat{k})^N}$$

$$\times\exp\left(\sum_i\sum_j(\hat{k}\cos\hat{\theta}_{0ij}-\hat{k}_i\cos\hat{\theta}_{1ij})\right) = L, \tag{44}$$

where $\hat{k}$ is the maximum likelihood estimator (mle) for κ under H_0, the $\hat{\theta}_{0ij}$ are the angles between the observations and the mle of the mean direction under H_0, the $\hat{k}_i$ are the mles for the κ_i under the H_1 and the $\hat{\theta}_{1ij}$ are the angles between the observations and the mle of the mean direction under H_1.

5.1.1 *Determining the mles under* H_0

The solution is well known, but for comparison with the next section it is repeated. The natural logarithm of the likelihood function, $l_0 = \ln(L_0)$, is

$$l_0 = N\ln(\kappa) - N\ln(\sinh\kappa) + \kappa\,\Sigma(p_iR_{x_i} + q_iR_{y_i} + r_iR_{z_i}) + \text{constant} \tag{45}$$

where R_{x_i}, R_{y_i} and R_{z_i} are the components of R_i. From the partial differentials of l_0, the joint mles of the p_i, q_i, r_i and κ are determined by solving the $(3m + 1)$ equations

$$\begin{aligned}&\hat{k}R_{x_i} - \lambda_i\hat{p}_i = 0\\&\hat{k}R_{y_i} - \lambda_i\hat{q}_i = 0\\&\hat{k}R_{z_i} - \lambda_i\hat{r}_i = 0 \qquad (i = 1,\ldots,m)\\&(N/\hat{k}) - \mathrm{N}\coth\hat{k} + \sum_i(\hat{p}_iR_{x_i} + \hat{q}_iR_{y_i} + \hat{r}_iR_{z_i}) = 0\end{aligned} \tag{46}$$

subject to the m constraints

$$\hat{p}_i^2 + \hat{q}_i^2 + \hat{r}_i^2 = 1, \tag{46a}$$

where the λ_i are Lagrange undetermined multipliers. The mles then satisfy

$$\coth\hat{k} - (1/\hat{k}) = \frac{\Sigma_iR_i}{N} \tag{47}$$

and

$$\sum_j\cos\theta_{0ij} = R_i. \tag{48}$$

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5.1.2 Determining the mles under H_1

Here the natural logarithm of the likelihood function is

$$l_1 = \sum_i N_i \ln(\kappa_i) - \sum_i N_i \ln(\sinh \kappa_i) + \sum_i \kappa_i (p_i R_{x_i} + q_i R_{y_i} + r_i R_{z_i}) + \text{constant}. \tag{49}$$

The joint mles are now determined by solving the $4m$ equations

$$\begin{aligned} &\hat{k}_i R_{x_i} - \lambda_i \hat{p}_i = 0 \\ &\hat{k}_i R_{y_i} - \lambda_i \hat{q}_i = 0 \\ &\hat{k}_i R_{z_i} - \lambda_i \hat{r}_i = 0 \qquad (i = 1, \ldots, m) \\ &(N_i/\hat{k}_i) - N_i \coth \hat{k}_i + \hat{p}_i R_{x_i} + \hat{q}_i R_{y_i} + \hat{r}_i R_{z_i} = 0 \end{aligned} \tag{50}$$

subject to the m constraints

$$\hat{p}_i^2 + \hat{q}_i^2 + \hat{r}_i^2 + 1. \tag{50a}$$

Hence, under H_1, the mles satisfy

$$\coth \hat{k}_i - (1/\hat{k}_i) = \frac{R_i}{N_i} \tag{51}$$

and

$$\sum_j \cos \hat{\theta}_{1\,ij} = R_i. \tag{52}$$

It should be noted that (52) is identical to (48).

5.1.3 Approximation to L for large precisions

The exponential term in the likelihood ratio (equation 44) is, from (52) and (48), given most simply by

$$\exp\left(\sum_i \sum_j (\hat{k} \cos \hat{\theta}_{0\,ij} - \hat{k}_i \cos \hat{\theta}_{1\,ij})\right) = \exp\left(\sum_i (\hat{k} - \hat{k}_i) R_i\right). \tag{53}$$

For t greater than 3 the approximations

$$2 \sinh t \cong \exp(t) \quad \text{and} \quad \coth t \cong 1 \tag{54}$$

are acceptably accurate. Hence if all the $\hat{k}_i$ exceed 3, using (47) and (51) the likelihood ratio reduces to

$$L = \frac{\hat{k}^N}{\hat{k}_1^{N_1}\, \hat{k}_2^{N_2} \ldots \hat{k}^{N_m}} \tag{55}$$

with

$$\hat{k} = \frac{N}{N - \Sigma_i R_i} \tag{56}$$

and

$$\hat{k}_i = \frac{N_i}{N_i - R_i}. \tag{57}$$

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Wilks (1938) has shown that $-2\ln L$ is asymptotically chi-square distributed with degrees of freedom given by the number of parameters under H_1 (i.e. $3m$) less the number of parameters under H_0 (i.e. $(2m+1)$). Hence

$$-2\ln L \sim \chi^2_{(m-1)}, \tag{58}$$

leading to the final equation

$$2\sum_i N_i \ln \hat{k}_i - 2N\ln \hat{k} = E \sim \chi^2_{(m-1)}. \tag{59}$$

Hence the null hypothesis H_0 may be rejected at the required level of significance if the observed value of E exceeds the critical value of the chi-square distribution.

5.2 BARTLETT'S METHOD

The test presented in Section 5.1 suffers from the problem that the derived statistic is only asymptotically chi-square distributed, thus requiring large samples for accurate application. Bartlett (1937) has presented a test which overcomes this problem for normally distributed variates.

It was noted above that

$$\frac{1}{k_i} = \frac{N_i - R_i}{N_i - 1} \tag{24}$$

is an unbiased estimate of $(1/\kappa_i)$. From (21) the distribution of $(1/k_i)$ is given by

$$\frac{1}{k_i} = \frac{S_i}{2\kappa_i(N_i-1)}, \quad \text{where} \quad S_i \sim \chi^2_{2(N_i-1)}. \tag{60}$$

Thus the approximate distribution of $(1/k_i)$ is the same as the distribution of an unbiased estimate for the variance of normally distributed variates, the variance being $(1/\kappa_i)$ with $2(N_i-1)$ degrees of freedom. Hence the analysis of Bartlett (1937) may be followed exactly, simply substituting $(1/\kappa_i)$ for the variance, $(1/k_i)$ for the unbiased estimate of the variance and $2(N_i-1)$ for the degrees of freedom. This will now be done.

5.2.1 Pooled, unbiased estimate for $(1/\kappa)$

Before using the test given by Bartlett it is necessary to have an unbiased, pooled estimate for $(1/\kappa)$, independent of the individual true mean directions. From (60)

$$\frac{N_i-1}{k_i} = \frac{S_i}{2\kappa}, \quad \text{where} \quad S_i \sim \chi^2_{2(N_i-1)}, \tag{61}$$

and therefore

$$\sum \frac{N_i-1}{k_i} = \frac{\Sigma S_i}{2\kappa} = \frac{U}{2\kappa}, \quad \text{where} \quad U \sim \chi^2_{2(N-m)}. \tag{62}$$

Hence, using $E(t)$ to denote the expectation value of t,

$$E\left(\frac{\Sigma(N_i-1)/k_i}{(N-m)}\right) = \frac{E(U)}{2\kappa(N-m)} = \frac{1}{\kappa}. \tag{63}$$

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Thus the statistic $(1/K)$, given by

$$\frac{1}{K}=\frac{\Sigma(N_i-1)/k_i}{(N-m)}=\frac{\Sigma(N_i-R_i)}{(N-m)}, \tag{64}$$

is the required pooled, unbiased estimate for $(1/\kappa)$. It should be noted that this estimate is independent of a common true mean direction and dependent only on the hypothesis of a common precision. This estimate has been (implicitly) used in the tests (25) and (41).

5.2.2 *Use of Bartlett's test*

Following Bartlett's analysis

$$\frac{2}{C}\ \{\Sigma(N_i-1)\ \ln k_i-(N-m)\ln K\}\ =G\sim\chi^2_{(m-1)} \tag{65}$$

where

$$C=1+\frac{1}{6(m-1)}\left\{\Sigma\,\frac{1}{N_i-1}-\frac{1}{N-m}\right\}. \tag{66}$$

Hence the null hypothesis of a common precision parameter may be rejected at the required level of significance if the observed value of G exceeds the critical value of the chi-square distribution.

The test statistics E proposed in (59) used maximum likelihood estimators together with numbers of observations; it is only asymptotically distributed as χ^2. The statistic G of (65) is based on Bartlett's suggestion of using degrees of freedom rather than numbers of observations and using unbiased estimators rather than maximum likelihood estimators; he also introduced the factor C to make the distribution more nearly χ^2. Monte Carlo experiments indicate that (65) is in fact applicable for values of N_i as small as 2 or 3. It should be noted, however, that this test is sensitive to deviations from a Fisher distribution and consequently in marginal cases it is probably wise to invest the effort of testing the precision in pairs.

Stephens (1969, section 5.3) and Mardia (1972, equation 9.5.11)) both apply Bartlett's method to this situation, but unfortunately there are algebraic errors in their results.

6 Conclusions

The derivation of a conditional test for the discrimination of mean directions from Fisher distributions has been given. In the derivation there are several useful relations and, to aid the investigator who is interested merely in applying the test, these are repeated below.

If two sample mean directions (vector resultants of lengths R_1 and R_2) are drawn from populations sharing a common true mean direction then the approximate sampling distribution of the angle γ between these two directions, conditional on the observed values of R_1 and R_2, is

$$\kappa'\gamma^2\sim\chi^2_2$$

where

$$\kappa'=\frac{\kappa_1\kappa_2R_1R_2}{\kappa_1R_1+\kappa_2R_2}.$$

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If $\kappa_1 = \kappa_2$ then this reduces to

$$\kappa\left(R_1 + R_2 - \frac{R^2}{R_1 + R_2}\right) \sim \chi_2^2$$

which effectively gives the distribution of R conditional on R_1 and R_2. Unfortunately κ is rarely known and so the distribution of R conditional on R_1 and R_2, but independent of κ is required. This distribution has been derived for the general case (i.e. κ_1 not necessarily equal to κ_2) and the following two tests result.

For two populations having a common precision parameter the null hypothesis that these populations share a common true mean direction may be rejected at the level of significance p if

$$\frac{(R_1 + R_2 - R^2/(R_1 + R_2))}{2(N - R_1 - R_2)} > \left(\frac{1}{p}\right)^{1/(N-2)} - 1.$$

If the two populations do not have a common precision parameter the null hypothesis that these populations share a common true mean direction may be rejected at the level of significance p if

$$\frac{r((R_1 + R_2)^2 - R^2)}{2((N_1 - R_1) + r(N_2 - R_2))(R_1 + rR_2)} > \left(\frac{1}{p}\right)^{1/(N-2)} - 1,$$

where $r = \kappa_2/\kappa_1$. If, as is usually the case, the ratio r is unknown, it may be estimated by the ratio k_2/k_1, with $k_i = (N_i - 1)/(N_i - R_i)$. However, it must be recognised that in such a situation the test is only approximate and so a decision should be deferred if the significance of the test statistic is marginal.

An extension of the analysis shows that for m populations sharing a common true mean direction and a common precision parameter, the distribution of R, conditional on R_1, $R_2, \ldots, R_m$, but independent of κ, is

$$\frac{(N-m)}{(m-1)} \cdot \frac{\Sigma R_i - R^2/\Sigma R_i}{2(N - \Sigma R_i)} = g \sim F[2(m-1), 2(N-m)].$$

Thus the null hypothesis of a common true mean direction may be rejected if the observed value of g exceeds the critical value of the F distribution at the required level of significance.

The tests presented here are conditional upon the observed values of the R_i and must be interpreted as such.

If the m populations share a common precision parameter then

$$\frac{1}{K} = \frac{\Sigma(N_i - R_i)}{(N-m)}$$

is the pooled, unbiased estimate for $(1/\kappa)$. Further,

$$\frac{2}{C}\left\{\Sigma(N_i - 1)\ln(k_i) - (N-m)\ln(K)\right\} = G \sim \chi^2_{(m-1)}$$

where

$$C = 1 + \frac{1}{6(m-1)}\left\{\Sigma\frac{1}{N_i - 1} - \frac{1}{N-m}\right\}.$$

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The null hypothesis of a common precision parameter may then be rejected if the observed value of G exceeds the critical value of the chi-square distribution at the required level of significance. Monte Carlo experiments indicate that this test is applicable for values of N_i as small as 2 or 3.

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Appendix A: estimation of $r = \kappa_2/\kappa_1$

All that is necessary in the estimation of r is that the distribution of the estimate be known. Given that this distribution is known, confidence limits may be placed on the actual value of the test statistic (i.e. the left side of equation 33). Naturally it is philosophically appealing (but unnecessary) to have an estimate for r which gives an unbiased estimate for the test statistic. Unfortunately this leads to a complicated expression and so it is preferable to use a simpler estimate.

The next most appealing estimate is an unbiased estimate of r itself. Defining K_i as

$$K_i = \frac{N_i - 2}{N_i - R_i} \tag{A1}$$

it follows from equation (21) that

$$\frac{2\kappa_i(N_i - 2)}{K_i} \sim \chi^2_{2(N_i - 1)} \tag{A2}$$

and K_i is an unbiased estimated of κ_i (Mardia 1972; McFadden 1980). Defining r' as

$$r' = \frac{K_2}{K_1} \tag{A3}$$

it follows that

$$\frac{r'}{r} \cdot \frac{N_2 - 1}{N_2 - 2} \sim F[2(N_1 - 1), 2(N_2 - 1)] \tag{A4}$$

and so r' is an unbiased estimate of $r(= \kappa_2/\kappa_1)$. The distribution of r' is simple and so r' is an acceptable estimator for r. However, the values of k_1 and k_2 will probably have been calculated already for other purposes and the distribution of $r_e(= k_2/k_1)$ is even simpler than that for r'; from (34) this distribution is

$$\frac{r_e}{r} \sim F[2(N_1-1), 2(N_2-1)]. \tag{A5}$$

Although r_e is a biased estimate of r it is the estimator with the simplest possible distribution and so it is suggested that r_e, rather than r', be used as an estimator for r.

Appendix B: confidence limits for the test (33) when $\kappa_1 \neq \kappa_2$

Consider for example two mean directions obtained from two samples having the following observed statistics:

$N_1 = 26,\quad R_1 = 23.5,\quad k_1 = 10.0;$

$N_2 = 30,\quad R_2 = 28.4,\quad k_2 = 18.125.$

The estimated value of r, r_e, is then 1.8125 and from the distribution (A5) r lies between 1.061 and 3.129 with 95 per cent confidence. Using the given values, from (33) the null hypothesis of a common true mean direction may be rejected at the 95 per cent of confidence if

$$\frac{r(2693.61 - R^2)}{2(2.5 + 1.6r)(23.5 + 28.4r)} > 0.0507. \tag{B1}$$

Using the 95 per cent confidence limits for r, the value of R at which the null hypothesis should be rejected lies between 51.64 and 51.69 with 95 per cent confidence. Alternatively this may be stated by saying that the angle between the two observed mean directions at which the null hypothesis should be rejected lies between 10.3° and 11.6°. Using the estimate r_e of r this angle is 10.6°. Hence the uncertainty in the value of r leads to a maximum error of only 1° in this instance. The angle was in fact 13°, and the null hypothesis is to be rejected, even though there is some overlap of the 95 per cent confidence circles.

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