Dispersion on a sphere

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Any topological framework requires the development of a theory of errors of characteristic and appropriate mathematical form. The paper develops a form of theory which appears to be appropriate to measurements of position on a sphere.

The primary problems of estimation as applied to the true direction, and the precision of observations, are discussed in the subcases which arise. The simultaneous distribution of the amplitude and direction of the vector sum of a number of random unit vectors of given precision, is demonstrated. From this is derived the test of significance appropriate to a worker whose knowledge of precision lies entirely in the internal evidence of the sample. This is the analogue of 'Student's' test in the Gaussian theory of errors.

The general formulae obtained are illustrated using measurements of the direction of remanent magnetization in the directly and inversely magnetized lava flows obtained in Iceland by Mr J. Hospers.

1. Introduction

The theory of errors was developed by Gauss primarily in relation to the needs of astronomers and surveyors, making rather accurate angular measurements. Because of this accuracy it was appropriate to develop the theory in relation to an infinite linear continuum, or, as multivariate errors came into view, to a Euclidean space of the required dimensionality. The actual topological framework of such measurements, the surface of a sphere, is ignored in the theory as developed, with a certain gain in simplicity.

It is, therefore, of some little mathematical interest to consider how the theory would have had to be developed if the observations under discussion had in fact involved errors so large that the actual topology had had to be taken into account. The question is not, however, entirely academic, for there are in nature vectors with such large natural dispersions. The remanent magnetism found in igneous and sedimentary rocks, for example, either by reason of heterogeneity of conditions, or composition, at the time of formation, or of changes induced *in situ* since that time, or of disturbances due to the treatment of the specimens, do show such considerable dispersion that an adequate theory for the combination of such observations is now needed. Finally, it is the opinion of the author that certain misapprehensions as to the nature of inductive inference have arisen in examples drawn from the theory of the normal distribution, by reason of the peculiar characteristics of that distribution, and that the examination of these questions, in an analogous though analytically different situation, will exhibit them in a clearer light.

1·1. *The fundamental distribution*

We may take as our fundamental distribution of elementary errors over the surface of the unit sphere, which is the field of possible observations, that in which the frequency density is proportional to

$$e^{\kappa \cos\theta},$$

where θ is the angular displacement from the true position, at which $\theta = 0$ and the density is a maximum, provided κ is positive. κ is evidently a measure of precision.

When κ is large the distribution is effectively confined to a small portion of the sphere in the neighbourhood of the maximum, and the distribution tends to conform to a two-dimensional isotropic Gaussian distribution in which κ is the invariance, or the reciprocal of the variance, in all directions. When κ is zero, the distribution is uniform all over the spherical surface. More usually, however, there is a maximum density in the true direction, and a minimum in the opposite direction, in the neighbourhood of the antipole.

Since the area within limits $\mathrm{d}\theta$ is proportional to $\sin\theta\,\mathrm{d}\theta$ or to $-\mathrm{d}(\cos\theta)$, and the integral

$$\int_{-1}^{1} \mathrm{e}^{\kappa\cos\theta}\,\mathrm{d}(\cos\theta) = \frac{1}{\kappa}(\mathrm{e}^{\kappa}-\mathrm{e}^{-\kappa}),$$

the absolute element of frequency is found to be

$$\mathrm{d}f = \frac{\kappa}{2\sinh\kappa}\,\mathrm{e}^{\kappa\cos\theta}\sin\theta\,\mathrm{d}\theta.$$

Given a sample of points dispersed from a common centre, the estimate of the position of this centre having the greatest likelihood is found by making $S(\cos\theta)$ a maximum, where S stands for summation over the sample.

If λ, μ, ν are the direction cosines, relative to any axes, of the estimated direction from the centre of the sphere, and l, m, n those of any observed point, it is necessary to vary λ, μ, ν so as to maximize

$$S(l\lambda+m\mu+n\nu) = \lambda S(l)+\mu S(m)+\nu S(n),$$

as is readily done by taking $\dfrac{\lambda}{S(l)} = \dfrac{\mu}{S(m)} = \dfrac{\nu}{S(n)}$.

This affords a unique solution provided $S(l)$, $S(m)$ and $S(n)$ are not all zero. In any case the maximum, R, is given simply by

$$R^2 = S^2(l)+S^2(m)+S^2(n).$$

The estimated direction is thus that of the vector sum of unit vectors having the directions of the several observations, the vector sum having length R.

2. Distributions providing estimates of precision

Before considering the estimates of precision in the more realistic case in which neither the pole nor the precision is known *a priori*, it will be useful to consider the more abstract cases in which (1) the pole, or (2) the axis of the pole is given.

2·1. *Known pole*

If
$$\mathrm{d}f = \frac{\kappa}{2\sinh\kappa}\,\mathrm{e}^{\kappa\cos\theta}\sin\theta\,\mathrm{d}\theta,$$

and if x is the sum of N independent values of $\cos\theta$ the distribution of x will be given by

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^N \frac{\mathrm{e}^{\kappa x}\,\mathrm{d}x}{(N-1)!}\left\{(N-x)^{N-1}-N(N-2-x)^{N-1}+\ldots\right.$$
$$\left.+(-)^r\frac{N!}{(N-r)!\,r!}(N-2r-x)^{N-1}\right\},$$

where r is the largest integer less than $\frac{1}{2}(N-x)$. The derivation of the factor independent of κ is discussed in § 2·31.

A sufficient estimate of κ is given by

$$\coth k - \frac{1}{k} = \frac{x}{N}, \quad \text{the equation of maximum likelihood.}$$

The logarithmic likelihood of κ is expressible as

$$L = N\left\{\ln \kappa - \ln \sinh \kappa + \kappa\left(\coth k - \frac{1}{k}\right)\right\},$$

which involves κ and k only, so demonstrating sufficiency, and is maximal when $\kappa = k$; the amount of information is

$$I_\kappa = N\left(\frac{1}{\kappa^2} - \operatorname{cosech}^2 \kappa\right),$$

having a maximum of $\frac{1}{3}N$ when $\kappa = 0$.

2·2. *Known axis*

For a given pole, x (and k) may be either positive or negative, but negative values may be interpreted as positive values referred to the antipole. If, therefore, not the pole but only the axis is given, x may be taken to be always positive.

The distribution of x will then be

$$\frac{2}{(N-1)!}\left(\frac{\kappa}{2\sinh \kappa}\right)^N \cosh \kappa x\left\{(N-x)^{N-1} - N(N-2-x)^{N-1} + \dots + (-)^r \frac{N!}{(N-r)!\,r!}(N-2r-x)^{N-1}\right\}.$$

The logarithmic likelihood of κ is now

$$L = -N(\ln \sinh \kappa - \ln \kappa) + \ln \cosh \kappa x,$$

the score, which equated to zero gives the equation of estimation, is

$$\partial L/\partial \kappa = -N(\coth \kappa - 1/\kappa) + x \tanh \kappa x,$$

and the amount of information is

$$-\partial^2 L/\partial \kappa^2 = N(1/\kappa^2 - \operatorname{cosech}^2 \kappa) - x^2 \operatorname{sech}^2 \kappa x,$$

which does, indeed, depend only on κ and k, but cannot be so expressed explicitly.

2·31. *The topological factor*

The general distribution of R derived from a sample of N observations, and the simultaneous distribution of R and c, where c is the cosine of the angle of error, contain a function of R depending on the number of observations, which is of purely topological origin, and being somewhat complicated, may be elucidated by a preliminary enquiry.

The distribution of the sum of N deviates each distributed from -1 to $+1$ in the 'rectangular' distribution $\quad df = \frac{1}{2}dx \quad (-1 < x < 1)$,

was early discussed by Irwin (1927) and by Hall (1927). Its nature may be simply apprehended by the method developed by Fisher (1929) using the circumstance that sections of a hypercube of side 2, and in N dimensions, if taken normally to a diagonal, will have 'areas' proportional to the elementary frequencies of this distribution. Evidently, discontinuities occur at $N+1$ points, at intervals of two units each from $-N$ to $+N$. Between the discontinuities the distribution is of the form

$$f(x)\,\mathrm{d}x,$$

where f is a polynomial of degree $(N-1)$. At the discontinuities all differential coefficients down to the $(N-2)$th are continuous, so that the total frequency less than x is a polynomial of degree N with a discontinuity of the form

$$\alpha_i(x-k_i)^N$$

as the ith point of discontinuity is past.

We may now impose the conditions that the total frequency less than x is constant and equal to unity when $x>N$, while it is zero when $x<-N$; the coefficients necessary to satisfy these conditions are readily shown to be proportional to those of the expansion of

$$(1-1)^N,$$

so that
$$\alpha_i = \frac{1}{2^N N!}\frac{(-)^i N!}{i!\,(N-i)!} \quad (i = 0, 1, \ldots, N),$$

since, as the $N+1$ values k_i differ by equal steps of 2, we may use the fact that

$$\Delta_i^n(x-k_i)^N = 2^N N!$$

for all values of x.

The probability that the sum of the N components is less than x is therefore

$$\frac{1}{2^N N!}\left\{(x+N)^N - N(x+N-2)^N + \ldots + (-)^s \frac{N!}{s!\,(N-s)!}(x+N-2s)^N\right\},$$

where s is the largest integer in $\frac{1}{2}(x+N)$. Similarly, the proportion greater than x is

$$\frac{1}{2^N N!}\left\{(N-x)^N - N(N-z-2)^N + \ldots + (-)^s \frac{N!}{s!\,(N-s)!}(N-x-2s)^N\right\},$$

where s is the largest integer in $\frac{1}{2}(N-x)$.

These expressions are everywhere differentiable with respect to x, and the frequency element is therefore

$$F(x)\,\mathrm{d}x = \frac{\mathrm{d}x}{2^N(N-1)!}\left\{(N-x)^{N-1} - N(N-x-2)^{N-1} + \ldots + (-)^s \frac{N!}{s!\,(N-s)!}(N-x-2s)^{N-1}\right\}.$$

This may be recognized as the distribution of the projection on any arbitrarily chosen direction of the sum of N random unit vectors.

The distribution of the length R of the resultant of N random unit vectors is, rather curiously, easily derivable from that of its projection; for if we write the distribution of R from 0 to N in the form

$$f(R)\,\mathrm{d}R,$$

a randomly chosen direction will make with the vector sum an angle, of which the cosine has the distribution $\frac{1}{2}\mathrm{d}c$ between the limits 1 and -1. In the expression

$$f(R)\,\mathrm{d}R\,\tfrac{1}{2}\mathrm{d}c, \quad \text{we may substitute } \mathrm{d}c = \frac{1}{R}\,\mathrm{d}x,$$

and integrate with respect to R, if x is positive, from x to N. Hence

$$\int_x^N f(R)\frac{\mathrm{d}R}{2R} = F(x), \quad \frac{f(R)}{2R} = -\frac{\mathrm{d}}{\mathrm{d}R}F(R),$$

whence
$$f(R)\,\mathrm{d}R = \frac{2R\,\mathrm{d}R}{2^N(N-2)!}\left\{(N-R)^{N-2} - N(N-R-2)^{N-1} + \ldots + (-)^s\frac{N!}{s!\,(N-s)!}(N-R-2s)^{N-2}\right\},$$

where s is the greatest integer in $\frac{1}{2}(N-R)$.

If we write this expression as

$$\frac{1}{2^N}\phi_N(R)\,2R\,\mathrm{d}R,$$

it follows that the simultaneous distribution of R and c, where c is the cosine of the angle between the resultant and an arbitrary direction is

$$\frac{1}{2^N}\phi_N(R)\,R\,\mathrm{d}R\,\mathrm{d}c.$$

Moreover, if a set of N_1 unit vectors give R_1 and c_1 with this frequency distribution, and N_2 other unit vectors give R_2, c_2 with a similar distribution, then the simultaneous distribution of R, c corresponding with the resultant must be

$$\frac{1}{2^{N_1+N_2}}\phi_{N_1+N_2}(R)\,R\,\mathrm{d}R\,\mathrm{d}c.$$

2·32. *The distribution of the sum of two observational unit vectors*

The general case of N observations requires somewhat heavy analysis, and the general procedure may be illustrated more clearly with the case of two observations. In accordance with the form adopted in the introductory section, the simultaneous distribution of two observations may be specified by the simultaneous distribution of the cosines of their angular errors

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^2 e^{\kappa(c_1+c_2)}\,\mathrm{d}c_1\,\mathrm{d}c_2,$$

together with the angle ψ between the planes containing the angles of error, independently distributed so that

$$\mathrm{d}f = \frac{1}{2\pi}\,\mathrm{d}\psi.$$

If R is the amplitude of the vector sum and c the cosine of its angular error, we have to determine the simultaneous distribution of R and c, by means of the relations

$$Rc = c_1 + c_2,$$

$$\tfrac{1}{2}R^2 = 1 + c_1c_2 + s_1s_2\cos\psi,$$

$$s_1^2 + c_1^2 = 1 = s_2^2 + c_2^2.$$

If the two angles are held constant, the variations of ψ and R are connected by the equation
$$\mathrm{d}\psi = R\,\mathrm{d}R/\surd\{(1-c_1^2)(1-c_2^2)-(\tfrac{1}{2}R^2-1-c_1c_2)^2\},$$
the quadratic in c_1, c_2 under the radical may now be rewritten as
$$R^2(1-\tfrac{1}{4}R^2)-\tfrac{1}{4}R^2(c_1-c_2)^2-(1-\tfrac{1}{4}R^2)(c_1+c_2)^2,$$
being of the form
$$A-\tfrac{1}{4}R^2(c_1-c_2)^2.$$
Moreover, the element $\mathrm{d}c_1\,\mathrm{d}c_2$ may be rewritten
$$\mathrm{d}c_1\,\mathrm{d}c_2 = \tfrac{1}{2}\mathrm{d}(c_1+c_2)\,\mathrm{d}(c_1-c_2),$$
and the integral of
$$\tfrac{1}{2}\mathrm{d}(c_1-c_2)/\surd\{A-\tfrac{1}{4}R^2(c_1-c_2)^2\}$$
between the limits at which the quadratic vanishes, is simply π/R irrespective of the value of A. This integral is to be taken twice, and divided by 2π to reduce $\mathrm{d}\psi$ to a frequency element.

A single operation of integration has thus expressed the frequency element in the form
$$\left(\frac{\kappa}{2\sinh\kappa}\right)^2 \mathrm{e}^{\kappa(c_1+c_2)}\,R\,\mathrm{d}R\,\frac{\mathrm{d}(c_1+c_2)}{R},$$
in which, substituting $\quad Rc \quad$ for c_1+c_2,

and $\quad R\,\mathrm{d}c \quad$ for $\mathrm{d}(c_1+c_2)$,

we find the simultaneous distribution of R and c,
$$\mathrm{d}f = \left(\frac{\kappa}{2\sinh\kappa}\right)^2 \mathrm{e}^{\kappa Rc}\,R\,\mathrm{d}R\,\mathrm{d}c.$$

2·33. *Mathematical induction of the general distribution*

As a third step we shall show that if the pairs of variables R_1, c_1 and R_2, c_2 are distributed as
$$\left(\frac{\kappa}{2\sinh\kappa}\right)^{N_1} \mathrm{e}^{\kappa R_1c_1}\,\phi_{N_1}(R_1)\,R_1\,\mathrm{d}R_1\,\mathrm{d}c_1$$
and
$$\left(\frac{\kappa}{2\sinh\kappa}\right)^{N_2} \mathrm{e}^{\kappa R_2c_2}\,\phi_{N_2}(R_2)\,R_2\,\mathrm{d}R_2\,\mathrm{d}c_2$$
in planes making a random angle, ψ, then the resultant pair R, c will be distributed in a similar distribution, where
$$\phi_N(R) = \frac{1}{(N-2)!}\left\{(N-R)^{N-2}-N(N-R-2)^{N-2}+\dots\right.$$
$$\left.+(-)^s\frac{N!}{s!\,(N-s)!}(N-R-2s)^{N-2}\right\},$$
and s is the largest integer less than $\tfrac{1}{2}(N-R)$. The analysis may follow closely the path of § 2·32. The new variables are defined by
$$Rc = R_1c_1+R_2c_2,$$
$$R^2-R_1^2-R_2^2 = 2R_1R_2(c_1c_2-s_1s_2\cos\psi);$$

the random element $d\psi$ is expressed as

$$d\psi = R\,dR/\sqrt{\{R_1^2R_2^2(1-c_1^2)(1-c_2^2)-\tfrac{1}{4}(R^2-R_1^2-R_2^2-2R_1R_2c_1c_2)^2\}},$$

in which the quadratic in c_1, c_2 may be rewritten as

$$-\tfrac{1}{4}(R_1^2-R_2^2)^2+\tfrac{1}{2}R^2(R_1^2+R_2^2)-\tfrac{1}{4}R^4$$
$$-\frac{1}{4R^2}\{2R^2(R_1^2+R_2^2)-R^4-(R_1^2-R_2^2)^2\}(R_1c_1+R_2c_2)^2$$
$$-\frac{1}{4R^2}\{(R^2+R_1^2-R_2^2)R_1c_1-(R^2-R_1^2+R_2^2)R_2c_2\}^2.$$

Correspondingly, the element $dc_1\,dc_2$ is equivalent to

$$dc_1\,dc_2 = \frac{1}{2R_1R_2R^2}\,d\{(R^2+R_1^2-R_2^2)R_1c_1-(R^2-R_1^2+R_2^2)R_2c_2\}\,d(R_1c_1+R_2c_2).$$

Hence, on integration over the admissible range, we find

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^{N_1+N_2} e^{\kappa Rc}\,\frac{2R^2}{2R_1R_2R^2}\,\phi_{N_1}(R_1)\,R_1\,dR_1\,\phi_{N_2}(R_2)\,R_2\,dR_2\,R\,dc\,dR,$$

where we have now included contributions from all values of c_1, c_2 and ψ compatible with R_1, R_2 and with the desired variates R and c.

Removing the factors which cancel out, the distribution found is

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^{N_1+N_2} e^{\kappa Rc}R\,dc\,dR\iint\phi_{N_1}(R_1)\,\phi_{N_2}(R_2)\,dR_1\,dR_2,$$

the integral being taken over all values of R_1 from 0 to N_1 and of R_2 from 0 to N_2 compatible with their having a resultant of length R. This integral is independent of κ, and in the case $\kappa=0$, it has already been shown that the corresponding distribution is

$$(\tfrac{1}{2})^{N_1+N_2}\,\phi_{N_1+N_2}(R)\,R\,dR\,dc.$$

Hence, in general, the simultaneous distribution of R and c from N observations is found to be

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^{N} e^{\kappa Rc}\,\phi_N(R)\,R\,dR\,dc,$$

where $\phi_N(R)$ stands for

$$\frac{1}{(N-2)!}\left\{(N-R)^{N-2}-N(N-R-2)^{N-2}+\ldots+(-)^s\frac{N!}{s!\,(n-s)!}(N-R-2s)^{N-2}\right\},$$

using as many terms as are required.

3. The test of significance based on a homogeneous sample

It has been shown in § 2·33 that the simultaneous distribution of the length R and the error angle $\cos^{-1}c$, of the resultant of N observational unit vectors is given by

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^{N} e^{\kappa Rc}\,\phi_N(R)\,R\,dR\,dc,$$

where $\phi_N(R)$ is a function of R and N only.

It follows that the unconditional (or marginal) distribution of R, irrespective of the error angle, is

$$\left(\frac{\kappa}{2\sinh\kappa}\right)^N \frac{2\sinh(\kappa R)}{\kappa}\,\phi_N(R)\,\mathrm{d}R,$$

by integrating with respect to c between the limits ± 1; while that of c, conditional on given R is, by division,

$$\frac{\kappa R\,\mathrm{e}^{\kappa Rc}\,\mathrm{d}c}{2\sinh(\kappa R)}.$$

Since this distribution still involves the parameter κ, of which the exact value is unknown, and since R, c is an exhaustive set of statistics, we may, on the supposition that the only available information as to the value of κ is that provided by the sample, proceed to eliminate the unknown parameter from the test of significance, by the method first introduced by 'Student' (1908). The interest of doing so is not only to provide a test of significance of immediate utility, but to exhibit a form of argument which seems to have been a good deal misunderstood by later writers.

In the general treatment when κ is not so large that $\mathrm{e}^{-2\kappa}$ can be ignored, the analysis would be intricate. In the practical applications we have immediately in view, the values of κ of interest are amply large enough for the use of the limiting form. In many cases also, the value of R observed is such that $N-R$ does not exceed 2, so that only the first term appears in the factor $\phi_N(R)$. In such cases the unconditional distribution of R becomes

$$\mathrm{d}f = \frac{(N-R)^{N-2}}{(N-2)!}\,\kappa^{N-1}\,\mathrm{e}^{-\kappa(N-R)}\,\mathrm{d}R,$$

and the conditional distribution of c is

$$\mathrm{d}f = \kappa R\,\mathrm{e}^{-\kappa R(1-c)}\,\mathrm{d}c.$$

The first step in eliminating κ is to find the probability, given κ, that the variate shall exceed any given value R, namely,

$$P(R,\kappa) = 1-\mathrm{e}^{-\kappa(N-R)}\left\{1+\kappa(N-R)+\ldots+\frac{1}{(N-2)!}\kappa^{N-2}(N-R)^{N-2}\right\}.$$

This value tends to unity as $\kappa\to\infty$, and in the limiting form here used is zero when κ is zero. In general, it has a finite value at this limit, and fiducially κ has this finite probability of being actually zero. The fiducial distribution of κ above this value is given by

$$\frac{\partial}{\partial\kappa}P(R,\kappa)\,\mathrm{d}\kappa = \frac{1}{(N-2)!}\,\kappa^{N-2}(N-R)^{N-1}\,\mathrm{e}^{-\kappa(N-R)}\,\mathrm{d}\kappa.$$

We multiply this frequency element by the probability, given κ, that c falls in the range $\mathrm{d}c$, obtaining

$$\frac{1}{(N-2)!}\,R\,\mathrm{d}c(N-R)^{N-1}\,\kappa^{N-1}\,\mathrm{e}^{-\kappa(N-Rc)}\,\mathrm{d}\kappa,$$

of which the integral over all possible values of κ from 0 to ∞ is

$$(N-1)\,\frac{(N-R)^{N-1}}{(N-Rc)^N}\,R\,\mathrm{d}c,$$

being the probability distribution of the cosine of the angular error, when the unknown precision κ has been eliminated.

The probability of a cosine less than c is now seen to be simply

$$P = \left(\frac{N-R}{N-Rc}\right)^{N-1},$$

or, expressing c in terms of the probability,

$$1-c = \frac{N-R}{R}\left\{\left(\frac{1}{P}\right)^{1/(N-1)} - 1\right\}.$$

There is here no limitation on the size of N, which may be so small as 2, in the case of greatest uncertainty as to the precision. When, however, N is large we may compare the value

$$\left(\frac{N-R}{N-Rc}\right)^{N-1} = \left(1+\frac{R(1-c)}{N-R}\right)^{-(N-1)}$$

with the conditional probability $\quad e^{-\kappa R(1-c)}$,

and note the equivalence of the former to that of accepting the estimate

$$k = \frac{N-1}{N-R},$$

the most likely value of κ given R. The small-sample solution, however, is not equivalent to the adoption of any estimate for κ, but takes account of the likelihood of all possible values.

These operations are heavier when R comes to be less than $N-2$, and further terms have to be taken in. The expression for the probability, calculated fiducially, that the angular error exceeds that of which the cosine is c, may however, be reduced to

$$P = \frac{1}{(N-Rc)^{N-1}}\left\{(N-R)^{N-1} - N\,\frac{R(1-c)}{R-Rc+2}\,(N-R-2)^{N-1} + \frac{N(N-1)}{2}\,\frac{R(1-c)}{R-Rc+4}\,(N-R-4)^{N-1} + \ldots\right\},$$

in which a new term is taken in as each discontinuity

$$R = N-2, N-4, N-6, \ldots$$

is passed.

4. Numerical examples

The test of significance developed in § 3 is, in its application, of extreme simplicity. My examples are drawn from the very fine body of data on the remanent magnetism of Icelandic lava flows, historic and prehistoric, put at my disposal by Mr J. Hospers of Pembroke College, Cambridge, and not yet published.

(*a*) From the recent lava flow of 1947–48, nine specimens gave the values for declination (measured from N. at 0° through E. at 90°), and inclination or dip, shown in table 1, in which are also shown the three direction cosines of each specimen, which form the basis of further calculations.

Table 1. Nine specimens from a recent lava flow

specimen number	decl.	incl.	direction cosines down	N.	E.
631	343·2	66·1	0·9143	0·3878	−0·1171
632	62·0	68·7	0·9317	0·1705	0·3207
633	36·9	70·1	0·9403	0·2722	0·2044
634	27·0	82·1	0·9905	0·1225	0·0624
635	359·0	79·5	0·9833	0·1822	−0·0032
636	5·7	73·0	0·9563	0·2909	0·0290
642 *A*	50·4	69·3	0·9354	0·2253	0·2724
643 *A*	357·6	58·8	0·8554	0·5176	−0·0217
644	44·0	51·4	0·7815	0·4488	0·4334
		totals	8·2887	2·6178	1·1803

The sums of the direction cosines of each kind are squared and added, to give the square of the resultant vector, of which the length, R, is found to be 8·77203. The direction cosines of the resultant are then

0·9449 0·2984 0·1346

To find the radius of confidence around this point, at the 5 % level, we have

$$R = 8{\cdot}77203, \quad N = 9, \quad N-R = 0{\cdot}22797.$$

Then

$$1/P = 20,$$
$$20^{\frac{1}{8}} = 1{\cdot}45422,$$
$$(N-R)\,(0{\cdot}45422) \div R = 0{\cdot}011804,$$
$$c = 0{\cdot}98820,$$
$$\theta = 8{\cdot}8^{\circ}.$$

So that possible directions more than 8·8° away from that indicated are excluded at the 5 % level of significance.

Estimation of the precision κ is not required for the test; we may, however, note that

$$\hat{\kappa} = (N-1)/(N-R)$$

is, in this case, 35·09, so that the neglect of terms in $e^{-2\kappa}$ is entirely appropriate.

(*b*) As a second example one is needed in which $N-R$ exceeds 2, as will readily occur if the homogeneity is lower, or the items more numerous. At levels ascribed to the early Quaternary, Hospers has found a considerable series of flows with reversed polarization. For forty-five specimens from these the sums of the direction cosines are found to be:

$$-37{\cdot}2193, \quad -11{\cdot}6127, \quad +0{\cdot}6710.$$

The length of the resultant vector is 38·9946; its direction cosines are:

down	N.	E.
−0·9545	−0·2978	+0·0172

almost diametrically opposite to the simple dipole field appropriate to the latitude which has components:

down	N.	E.
0·9724	0·2334	0

The cosine of the angle between these directions is therefore $-0{\cdot}99766$, making the angle with the antipodal direction only $3{\cdot}9°$.

Although $N-R$ is much greater than 2, the simple formula may be relied on to give a good first approximation to the radius of confidence:

$$20^{-44} = 1{\cdot}070456,$$

$$1-c = 0{\cdot}01085,$$

$$\theta = 8{\cdot}45°.$$

If this is a sufficiently good approximation, the mean of these somewhat variable directions, although fairly accurately determined, is not significantly different from the reversed dipole field. The case, however, is evidently one in which the contribution of terms beyond the first must be examined. With $N = 45$, $R = 38{\cdot}9946$, and the trial value $\theta = 8{\cdot}45°$, $c = 0{\cdot}9891445$, we have table 2.

TABLE 2. FORM OF COMPUTATION WHEN ANY TERM BEYOND THE FIRST IS TO BE CONSIDERED

	number	common log	
$N-Rc$	6·4287	0·8081232	
$N-R$	6·0054	0·7785419	
ratio		0·0295813	
$(\text{ratio})^{-44}$	0·04983	98·69788	first term
$N-R-2$	4·0054	0·6026459	
ratio		0·2054773	
$(\text{ratio})^{-44}$		90·95900	
N	45	1·65321	
$R-Rc$	0·42331	99·62666	
$R-Rc+2$	2·42331	$-0{\cdot}38441$	
	$-0{\cdot}0^{8}7$	91·85446	second term

Even in this case, therefore, the second term is only about one seven-millionth of the first, and is of no practical consequence. The existence of four terms is thus no sufficient reason for thinking the first to be inadequate as an approximation. The computation of such terms as set out above is not, however, particularly difficult.

Secular variation will introduce discrepancies among flows of different ages, nevertheless the increase of number of specimens from 9 to 45 has been sufficient to give the latter estimates somewhat the higher precision.

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