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Statistical analysis of palaeomagnetic inclination data

Randolph J. Enkin[1] and Geoffrey S. Watson[2]
[1] *Geological Survey of Canada-Pacific*, PO Box 6000, *Sidney, BC, Canada*, V8L 4B2, *Canada*
[2] *Department of Mathematics, Fine Hall, Princeton University, Princeton, NJ*, 08544-1000, *USA*

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SUMMARY

Palaeomagnetic studies on bore core or on tectonically disturbed localities often lose declination information, but the inclination still offers important palaeogeographic information. While the arithmetic mean of inclinations, $\bar{I}$, is a biased estimator, the bias is negligible with shallow data. Using co-inclination $\bar{\theta} = 90° - |\bar{I}|$ and precision $\kappa^* = 1/\text{variance}$, we find that the arithmetic mean and associated 95 per cent confidence interval are acceptable estimates when $\bar{\theta}\sqrt{\kappa^*} > 400°$. When inclination is steep and /or precision low, numerical methods must be applied. We develop the likelihood function for θ and κ and offer an efficient method to find its maximum, $(\hat{\theta}, \hat{\kappa})$, and to calculate the confidence interval. When $\hat{\theta}\sqrt{\hat{\kappa}} < 200°$, the confidence interval is asymmetric about the mean. When sites are collected from several rigid blocks, the relative declinations within each block can be useful. Using 'block-rotation Fisher analysis', better inclination estimates with tighter confidence intervals can be made, even on very steep data. We describe how to apply these methods to an inclination-only fold test. The techniques are illustrated on real data and are tested extensively using numerical simulations.

Key words: numerical techniques, palaeomagnetism.

INTRODUCTION

The time-averaged geomagnetic field is well approximated by a geocentric axial dipole, so that the mean inclination (I) is a simple function of the latitude of the measuring site. The goal of many palaeomagnetic studies is to determine the palaeolatitude (λ) of a site at the time the studied rock sequence was magnetized.

In the ideal case, there is sufficient geological information to restore samples to the orientations they had when they became magnetized. Sometimes these restorations cannot be made. Bore core, for example, is seldom azimuthally oriented, and, in orogenic belts, unknown vertical axis rotations can make it impossible to bring different sections into their original orientations. But so long as the attitude of the palaeohorizontal plane can be determined, rotation about the strike (i.e. simple horizontal-axis rotation) brings the inclination to its correct pre-tilt value, whatever the full structural correction might be; the measured palaeomagnetic declination may not be meaningful, but the inclination is well defined and may be used for tectonic reconstructions and other purposes.

Because of secular variation of the geomagnetic field and measuring uncertainties, palaeomagnetic directions must be correctly averaged together to produce a geologically significant interpretation. When measurements of both inclination and declination are available ('full data'), one applies the techniques introduced by Fisher (1953). However, if only inclination is available ('inclination-only data') it is necessary to consider its statistical distribution so that estimates of the mean inclination and its confidence interval can be made.

Since palaeomagnetic observations are distributed on a sphere, the arithmetic average of measured inclinations is biased towards shallow directions; that is, in any rotationally symmetrical distribution, there will be more inclinations shallower than the mean (nearer the equator) than steeper (nearer the pole) (Fig. 1a). The problem is aggravated when the mean is near vertical (high palaeolatitude) or when dispersion is large, because some directions 'overshoot' the pole, biasing downwards the calculated average (Fig. 1b). As pointed out by Briden & Ward (1966), if the true inclination is 90°, all measured directions will have shallower inclinations and thus the arithmetic average will be less than 90°.

The averaging of inclination-only data has been dealt with before (Briden & Ward 1966; Kono 1980; McFadden & Reid 1982; Clark & Morrison 1983; Clark 1983, 1988; Cox & Gordon 1984). These workers assume, as we do, that palaeomagnetic directions are distributed according to the Fisher (1953) distribution. Explicit estimates of the mean are not possible, and these workers obtain estimates using various analytical approximations and numerical solutions. These procedures differ considerably, but most yield similar estimates of mean inclination and its confidence interval. All authors agree that estimates are very good, except for near-vertical or highly dispersed data.

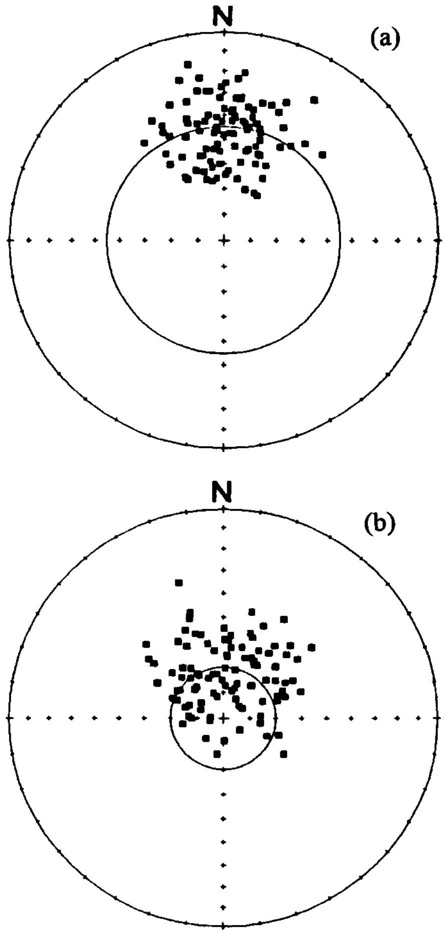


Figure 1. Stereographs of Fisher-distributed samples with $N = 100$ and $\kappa = 20$, where the mean inclination is 45° in (a) and 70° in (b). Circles are drawn at constant inclination equal to the mean. Note that the majority of the sample lies on the shallow side of the circle, indicating that an arithmetic mean of the inclinations will be biased shallow. When the mean inclination is steep, as in (b), some directions in the sample overshoot the pole, rendering an even shallower arithmetic mean.

The estimates we present in this paper differ in form from the others, but in most cases give similar results. We argue that this new formulation is conceptually simpler than previous ones and comparatively approximation-free. Our estimates are applicable to steep and highly dispersed data. Furthermore, we introduce a simple criterion to help the practitioner decide if the inclination is sufficiently shallow and tightly distributed to allow an accurate estimate using simply the arithmetic mean and standard error.

We proceed to show that the inclination-only problem is a special case of a more general problem where groups of directions are determined from rigid blocks that have suffered differential vertical-axis rotations. The resulting method, block-rotation Fisher analysis, gives more robust results and tighter confidence limits than the inclination-only method, especially for studies involving steep inclinations (very high palaeolatitude).

The fold test in palaeomagnetism is the most utilized field test to determine the stability of palaeomagnetic remanence. Even when sites have suffered relative vertical-axis rotations it is often still important to consider a fold test, so we introduce the use of the method in analysing data from tilted beds and set out procedures for carrying out an inclination-only fold test.

ESTIMATION OF THE INCLINATION

The marginal likelihood function for the inclination-only problem, given a Fisher-distributed sample, may be found in all the papers dealing with this subject, starting with Briden & Ward (1966). Rather than declination (D) and inclination (I), we use polar angle or co-inclination ($\theta = 90° - I$) and azimuth ($\phi = D$) to specify directions. If a set of directional data comes from a Fisher distribution with mean direction (θ_μ, ϕ_μ) and concentration κ, then the marginal distribution of θ (that is, the probability density function of θ alone) is

$$f(\theta) = \frac{\kappa \sin\theta}{4\pi \sinh\kappa} \times \int_0^{2\pi} \exp(\kappa[\cos\theta_\mu \cos\theta + \sin\theta_\mu \sin\theta \cos(\phi_\mu - \phi)])\, d\phi . \quad (1)$$

The curve of $f(\theta)$ between 0 and π has a peak near $\theta = \theta_\mu$, but is not symmetrical. The goal of this paper is to determine an accurate and practical estimate of θ_μ, the true co-inclination of the full Fisher-distributed data. The notation and solution of eq. (1) is simplified by using the modified Bessel function,

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(x \cos\phi)\, d\phi , \quad (2)$$

so (1) becomes

$$f(\theta) = \frac{\kappa \sin\theta}{2 \sinh\kappa} \exp(\kappa \cos\theta_\mu \cos\theta) I_0(\kappa \sin\theta_\mu \sin\theta) . \quad (3)$$

Graphs and formulae for the modified Bessel functions can be found in Abramowitz & Stegun (1964). For our calculations we need the power series,

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \cdots , \quad (4)$$

for small arguments, and the asymptotic expansion,

$$I_0(x) = \frac{\exp x}{\sqrt{2\pi x}} \left(1 + \frac{1}{8x} + \frac{9}{128x^2} + \cdots\right), \quad (5)$$

for large arguments. The function is fitted well by these three-term approximations when (4) switches over to (5) at $x = 1.5$. We also need

$$\frac{I_0'}{I_0}(x) \equiv \frac{I_0'(x)}{I_0(x)} = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{96} + \cdots \quad (6)$$

for $x \leq 1.5$, and

$$\frac{I_0'}{I_0}(x) = 1 - \frac{1}{2x} - \frac{1}{8x^2} + \cdots \quad (7)$$

for $x > 1.5$.

It is instructive to consider the form of (3) when the dispersion is small (κ large) and the mean inclination is not near the poles (θ_μ far from 0° and 180°). Small $\sin\theta$ will be rare, so the term $I_0(\kappa \sin\theta_\mu \sin\theta)$ that appears in (3) is, to good approximation, given by the first term of (5). The approximate form of the θ density (3) then becomes

$$f(\theta) \approx \frac{\kappa \sin\theta}{2\sinh\kappa} \frac{\exp(\kappa\cos(\theta_\mu - \theta))}{\sqrt{2\pi\kappa\sin\theta_\mu\sin\theta}}. \quad (8)$$

Further, when κ is large, θ is never far from θ_μ, so we can approximate (8), using radians for angular measurements, by

$$f(\theta) \approx \sqrt{\frac{\kappa}{2\pi}} \exp\left(-\frac{\kappa}{2}(\theta_\mu - \theta)^2\right), \quad (9)$$

a Gaussian distribution about θ_μ with variance $1/\kappa$. Thus, when κ is large and θ_μ is away from 0° and 180°, if we were given a sample $(\theta_1, \ldots, \theta_N)$ from the density function (3), we could use the arithmetic mean,

$$\bar{\theta} \equiv \sum \theta_i / N, \quad (10)$$

as an estimator of θ_μ, and

$$\kappa^* \equiv 1/s_\theta^2 = (N-1) \Big/ \sum (\theta_i - \bar{\theta})^2 \quad (11)$$

as an estimator of κ, where s_θ^2 is the variance of the θs and angles are measured in radians. The 95 per cent confidence interval is given by

$$\bar{\theta} \pm \alpha_{95}^* \equiv \bar{\theta} \pm t_{0.975} s_\theta / \sqrt{N}, \quad (12)$$

where $t_{0.975}$ is the 97.5 per cent of the Student distribution with $N-1$ degrees of freedom. It would be more rational to write δ_{95} instead of α_{95}, since statisticians prefer to reserve α for probabilities, but we adhere to the established notation here and below. We call $\bar{\theta}$, κ^* and α_{95}^* the 'first-order estimators'. Clarke & Morrison (1983) discuss applications of this approximation.

Like McFadden & Reid (1982) and Clark (1983), we examined the use of maximum likelihood to define the optimal estimators of the parameters in (3), but we use rather more direct methods of maximizing the likelihood of the inclination data, which in our notation is given by

$$L = \prod_{i=1}^{N} f(\theta_i) = \left(\frac{\kappa}{2\sinh\kappa}\right)^N \prod_{i=1}^{N} \sin\theta_i \exp(\kappa\cos\theta_\mu\cos\theta_i) I_0(\kappa\sin\theta_\mu\sin\theta_i). \quad (13)$$

We attempted no correction for bias as McFadden & Reid (1982) did. Our simulations showed that their 'unbiased' estimate is usually within 0.1° of the arithmetic mean of the inclinations, and is thus as badly biased as $\bar{\theta}$.

When performing statistical operations in Cartesian coordinates where parameters can take any value, no special factor must be included in the likelihood function to take into account the prior information available to parameter estimation. In our case, however, the parameters we wish to fit do not sit in a flat space. We must consider the distortion due to spherical geometry and the constraint that $\kappa > 0$. One convenient way of doing this is by means of Bayes' rule, in which our prior information about the unknown parameters is specified in terms of a density function. To avoid a subjective choice of the prior distribution, Jeffreys (1939) introduced a principle that is designed to find, in each problem, a prior distribution that corresponds to a totally 'open mind' about the values of the parameters. Such priors are often called 'non-informative'. In our case, it leads to a prior distribution of the true mean direction that is uniform on the unit sphere. The non-informative joint density of (θ_μ, ϕ_μ) is proportional to $\sin\theta_\mu$, because the area element on the sphere is $\sin\theta \, d\theta \, d\phi$. The $\sin\theta$ factor forces the likelihood of a vertical inclination to be zero, which is geometrically required. The parameter $1/\kappa$ is analogous to σ^2 in the normal distribution. Box & Tiao (1973) demonstrate that $\log\sigma^2$, and therefore $\log\kappa$, should be uniformly distributed on $(0, \infty)$, despite being an improper distribution (i.e. the prior density of κ, $1/\kappa$, cannot be integrated from 0 to ∞). Clark (1983) considered maximum-likelihood estimators using (13) but realized that the conditions that $\sin\theta_\mu > 0$ and $\kappa > 0$ had to be imposed as constraints on the solution. Using the non-informative prior, $\sin\theta_\mu/\kappa$, assures these conditions in a much more natural fashion, independent of the choice of coordinate system.

The product of the likelihood (13) and the non-informative prior is proportional to the joint posterior density of θ_μ and κ, which we call L_{Bayes},

$$L_{\text{Bayes}} = \left(\frac{\sin\theta_\mu}{\kappa}\right)\left(\frac{\kappa}{2\sinh\kappa}\right)^N \prod_{i=1}^{N} \sin\theta_i \exp(\kappa\cos\theta_\mu\cos\theta_i) I_0(\kappa\sin\theta_\mu\sin\theta_i). \quad (14)$$

Instead of maximizing L_{Bayes}, it is numerically better to maximize its logarithm,

$$\ell' = \log\left(\frac{\sin\theta_\mu}{\kappa}\right) + N\log\left(\frac{\kappa}{2\sinh\kappa}\right) + \sum_{i=1}^{N} \{\log(\sin\theta_i) + \kappa\cos\theta_\mu\cos\theta_i + \log[I_0(\kappa\sin\theta_\mu\sin\theta_i)]\}, \quad (15)$$

or, eliminating constant terms,

$$\ell = \log\left(\frac{\sin\theta_\mu}{\kappa}\right) + N\log\left(\frac{\kappa}{\sinh\kappa}\right) + \sum_{i=1}^{N} \{\kappa\cos\theta_\mu\cos\theta_i + \log[I_0(\kappa\sin\theta_\mu\sin\theta_i)]\}. \quad (16)$$

Except for the Bayesian term, $\log(\sin\theta_\mu/\kappa)$, this is the log likelihood for inclination-only data.

In order to understand the properties and complexities of the inclination-only problem it is highly instructive to consider the form of the likelihood function. In Fig. 2 we plot contours of the log-likelihood function (16), $\ell(\theta_\mu, \kappa)$, for a case where θ_μ and κ are large (Fig. 2a) and for a case where they are small (Fig. 2c). The form of the likelihood function does not follow a simple single-peaked symmetric form. There is a 'ridge' of likelihood for inclination towards the vertical ($\theta = 0$). McFadden & Reid (1982) state that the co-inclination estimate, $\hat{\theta}$, of the true co-inclination, θ_μ, is correlated with the precision estimate ($\hat{\kappa}$), but Fig. 2 shows that for steep inclinations $\hat{\kappa}$ is quite well resolved, independently of the value of $\hat{\theta}$.

Estimates of $\hat{\theta}$ and $\hat{\kappa}$ can be produced by locating the maximum of the Bayesian likelihood function. Note that we find standard optimization techniques, such as the conjugate gradient method, inefficient at locating this maximum, especially when there is a long ridge as in Fig. 2(c). We propose an iterative algorithm that is robust and efficient and tailored to our specific problem. At maximum, $\partial\ell/\partial\theta_\mu = 0$ and $\partial\ell/\partial\kappa = 0$, so we can find

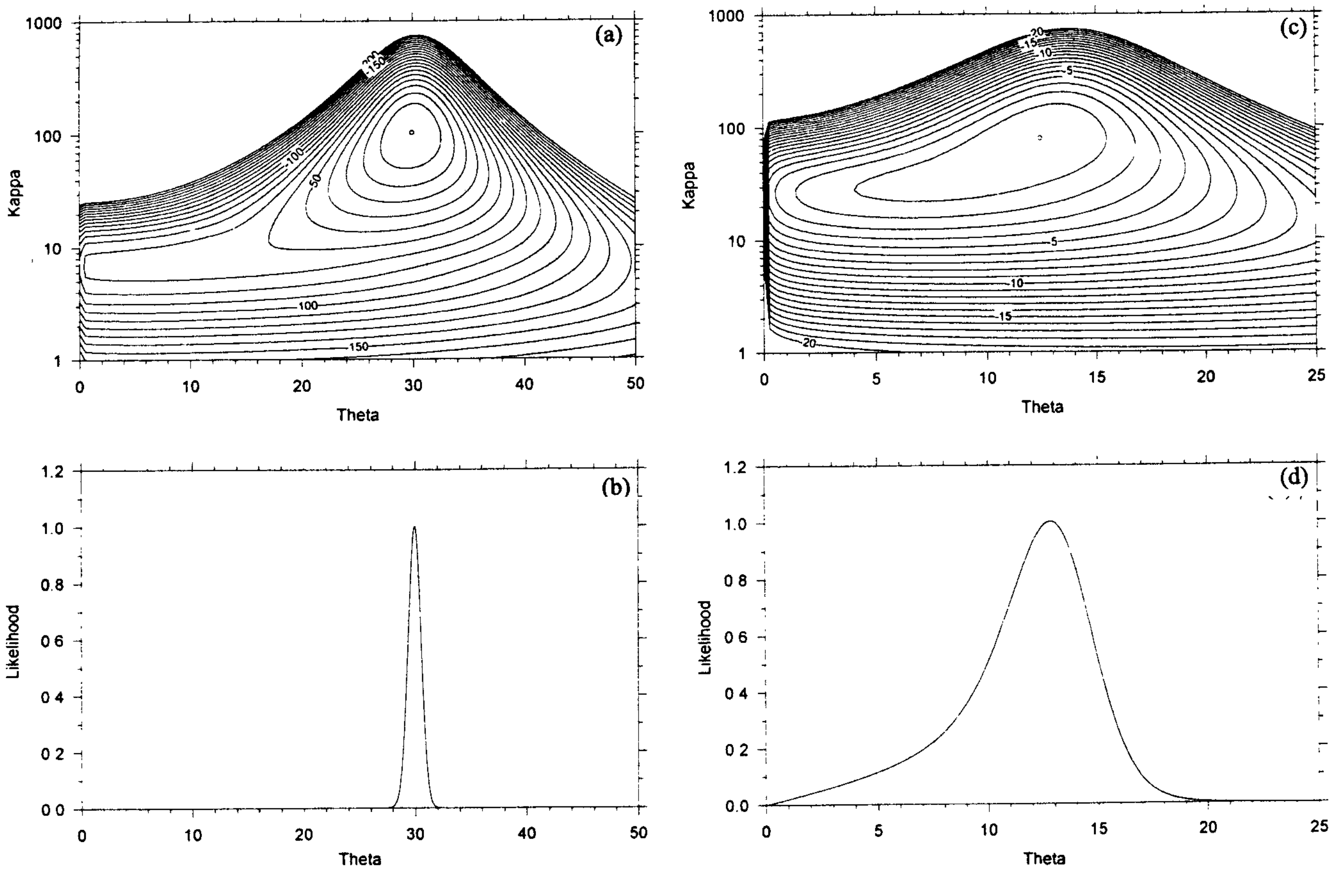


Figure 2. Plots of the inclination-only likelihood function for Fisher-distributed samples with (a), (b) $N = 100$, co-inclination $\theta_\mu = 30°$ and precision $\kappa = 100$ and (c), (d) $N = 10$, $\theta_\mu = 10°$ and $\kappa = 30$. (a) and (c) are contour plots of the log likelihood (16), while (b) and (d) are plots of the marginal likelihood of θ (21). All plots are normalized such that the maximum likelihood is set to 1. Note that the contours are not elliptical but rather have a pronounced ridge developed towards the vertical. When the data are shallow or concentrated enough, as in (a) and (b), this tail has negligible likelihood, so the likelihood of θ alone is well fitted by a Gaussian distribution ($\theta = 30.0° \pm 1.1°$). For steep or dispersed data, as in (c) and (d), the maximum and its associated confidence interval must be located numerically ($\hat{\theta} = 12.8°\,^{+4.2°}_{-8.4°}$).

analytical equations for $\hat{\theta}$ and $\hat{\kappa}$:

$$\cot\hat{\theta} + \sum_{i=1}^{N}\left\{-\hat{\kappa}\sin\hat{\theta}\cos\theta_i + \hat{\kappa}\cos\hat{\theta}\sin\theta_i\frac{I_0'}{I_0}(\hat{\kappa}\sin\hat{\theta}\sin\theta_i)\right\} = 0, \qquad (17)$$

$$-\frac{1}{\hat{\kappa}} + N\left(\frac{1}{\hat{\kappa}} - \coth\hat{\kappa}\right) + \sum_{i=1}^{N}\left\{\cos\hat{\theta}\cos\theta_i + \sin\hat{\theta}\sin\theta_i\frac{I_0'}{I_0}(\hat{\kappa}\sin\hat{\theta}\sin\theta_i)\right\} = 0. \qquad (18)$$

Hence

$$\tan\hat{\theta} = \left(\frac{1}{N}\sum_{i=1}^{N}\sin\theta_i\frac{I_0'}{I_0}(\hat{\kappa}\sin\hat{\theta}\sin\theta_i) + \frac{1}{N\hat{\kappa}\sin\hat{\theta}}\right)\Bigg/\left(\frac{1}{N}\sum_{i=1}^{N}\cos\theta_i\right), \qquad (19)$$

$$\hat{\kappa} = (N-1)\Bigg/\left(N - \sum_{i=1}^{N}\left\{\cos\hat{\theta}\cos\theta_i + \sin\hat{\theta}\sin\theta_i\frac{I_0'}{I_0}(\hat{\kappa}\sin\hat{\theta}\sin\theta_i)\right\}\right), \qquad (20)$$

where $\coth(\hat{\kappa})$ is taken to be 1 (a very good approximation for the palaeomagnetically reasonable condition that $\kappa > 3$). Note that the effect of the non-information term ($\sin\theta_\mu/\kappa$) becomes negligible when N becomes large. In (20) this term modifies N to $N-1$ in the numerator. We have divided each term in (19) by N to show that $\tan\hat{\theta}$ is a quotient of $\sin\theta_i$ and $\cos\theta_i$ averages with an added non-information term that tends to zero as N gets large. When $\hat{\kappa}\sin\hat{\theta}$ is sufficiently large, (19) and (20) reduce to the familiar equations $\tan\hat{\theta} = \Sigma\sin\theta_i/\Sigma\cos\theta_i$ and $\hat{\kappa} = (N-1)/(N-R)$, where R is the resultant length of all the sample unit vectors.

We determine the maximizing estimates by starting with the first-order estimators and then iteratively applying (19) and (20) until they converge. We find that it is necessary to use the first three terms of series expansions (6) and (7). McFadden & Reid (1982) essentially use only the first term of (5) in their development, which is a poor approximation for $x < 15$ (Clark 1983). In practice, three iterations usually resolve $\hat{\theta}$ to within 0.1°. When there is a pronounced ridge, however, the iteration approaches the maximum too slowly. Restarting the iteration at $\hat{\theta}/2$ and $\hat{\kappa}/2$ finds the maximum easily. Thus we routinely make two iterative attempts at finding the maximum and take the solution that corresponds to the higher likelihood.

As important as the point estimate of $\hat{\theta}$ is its associated confidence interval. With the Bayesian approach, the posterior density of θ_μ is found by integrating out κ in (14), as shown in Figs 2(b) and (d). Taking $L_2(\theta, \kappa) \equiv L_{\text{Bayes}}$ (14) we get the marginal posterior density of θ_μ:

$$L_1(\theta) = \int_0^\infty L_2(\theta, \kappa)\, d\kappa. \tag{21}$$

In practice, for each θ we numerically integrate (14) over $\kappa = 1$ to 1000 with κ steps spaced exponentially. The two-parameter maximum likelihood estimate for co-inclination, $\hat{\theta}_2$ (i.e. $\hat{\theta}$ from eq. 16), is steeper than the single-parameter likelihood estimate $\hat{\theta}_1$ determined by maximizing $L_1(\theta)$. This maximum must be located numerically.

When inclinations are shallow enough or precisions high enough, L_1 can be approximated well by a normal distribution with mean $\hat{\theta}_2$ and variance $1/N\hat{\kappa}$ (e.g. Fig. 2b). Thus the 95 per cent confidence limits are approximately two standard errors about the mean $[\pm(1.960/\sqrt{N\hat{\kappa}})180°/\pi]$. Note that our normal approximation is more accurate than that proposed by Clark & Morrison (1983), who use the first-order estimates (10) and (11) of θ and κ rather than their maximum likelihood estimates.

However, our normal approximation fails when some of the directions in the sample are close to vertical. In such cases (e.g. Fig. 2d) we must take an interval that contains 95 per cent of the posterior probability (L_1) and includes only values more likely than any outside it. Demarest (1983) considered the equivalent problem for the case when full data (inclination and declination) are available. To find the upper and lower 95 per cent confidence limits, he integrated the two tails of the marginal likelihood function to limits such that each contained 2.5 per cent of the total area. As we are integrating the marginal likelihood of θ, we need a range of integration that collects 95 per cent of the probability and includes only points where $L_1(\theta)$ is higher than any points outside the range, for we are trying to find a range for which θ is most likely. Because $L_1(\theta)$ has a single peak, the range is an interval (θ_L, θ_U), with L_1 taking equal values at each end. Thus we have the equations

$$L_1(\theta_L) = L_1(\theta_U), \tag{22}$$

$$\int_{\theta_L}^{\theta_U} L_1(\theta)\, d\theta \Big/ \int_{0°}^{180°} L_1(\theta)\, d\theta = 0.95, \tag{23}$$

where θ_L and θ_U are, repectively, the lower and upper limits. Defined as such, the confidence interval width is minimized, which is important in the case of skewed distributions, such as seen in Fig. 2(d). Note that the confidence limits are asymmetric about the maximum-likelihood estimate. For $\hat{\theta} < 90°$, we define $\alpha_{95}^+ \equiv \hat{\theta} - \theta_L$ and $\alpha_{95}^- \equiv \theta_U - \hat{\theta}$. When $\hat{\theta} > 90°$, the definitions of α_{95}^+ and α_{95}^- are reversed. The signs of these definitions are chosen to be compatible with inclination $I = 90° - \theta$.

Numerical determination of θ_L and θ_U is time-consuming, so we use a further approximation. As computers become faster such an approximation will not be necessary, so we will only sketch the method here. Assuming $0° < \hat{\theta}_1 < 90°$, the upper tail of L_1 is approximately Gaussian (as illustrated in Fig. 2d). The lower tail can be approximated as the sum of a straight line going through zero and a Gaussian. The slope of the straight line comes from the value of L_1 at 1°, the means of the Gaussians are set to the mode of L_1, and their standard deviations are calculated from the values of L_1 at $\hat{\theta}_1 \pm \alpha_{95}^*$. Once the parameters of the approximation are determined, θ_U and then the left-hand side of eq. (23) are easily evaluated functions of θ_L, which is varied until the confidence proportion [right-hand side of eq. (23)] is located. The only time-consuming step is locating $\hat{\theta}_1$, the position of the maximum of L_1.

As a worked example, consider the data chosen to produce Fig. 2(b): 76.7°, 75.6°, 74.9°, 86.7°, 68.2°, 71.7°, 69.5°, 80.5°, 76.1° and 81.0°. The mean inclination is 76.1° and the first-order estimator for $\kappa = 1/\text{variance} = 104.5$. The maximum likelihood estimates after five iterations of (19) and (20) are $\hat{\theta}_2 = 12.6°$ ($I = 77.4°$) and $\hat{\kappa} = 82.5$. A second iteration starting at $\theta = 6.3°$ and $\kappa = 41.2$ gives $\hat{\theta}_2 = 12.4°$ and $\hat{\kappa} = 76.7$ and a slightly higher likelihood, after 10 steps. The Gaussian estimate of inclination is thus $77.6° \pm 4.1°$. Using the marginal likelihood function, the estimate becomes $77.2^{+8.4}_{-4.2}$.

PRACTICAL CONSIDERATIONS

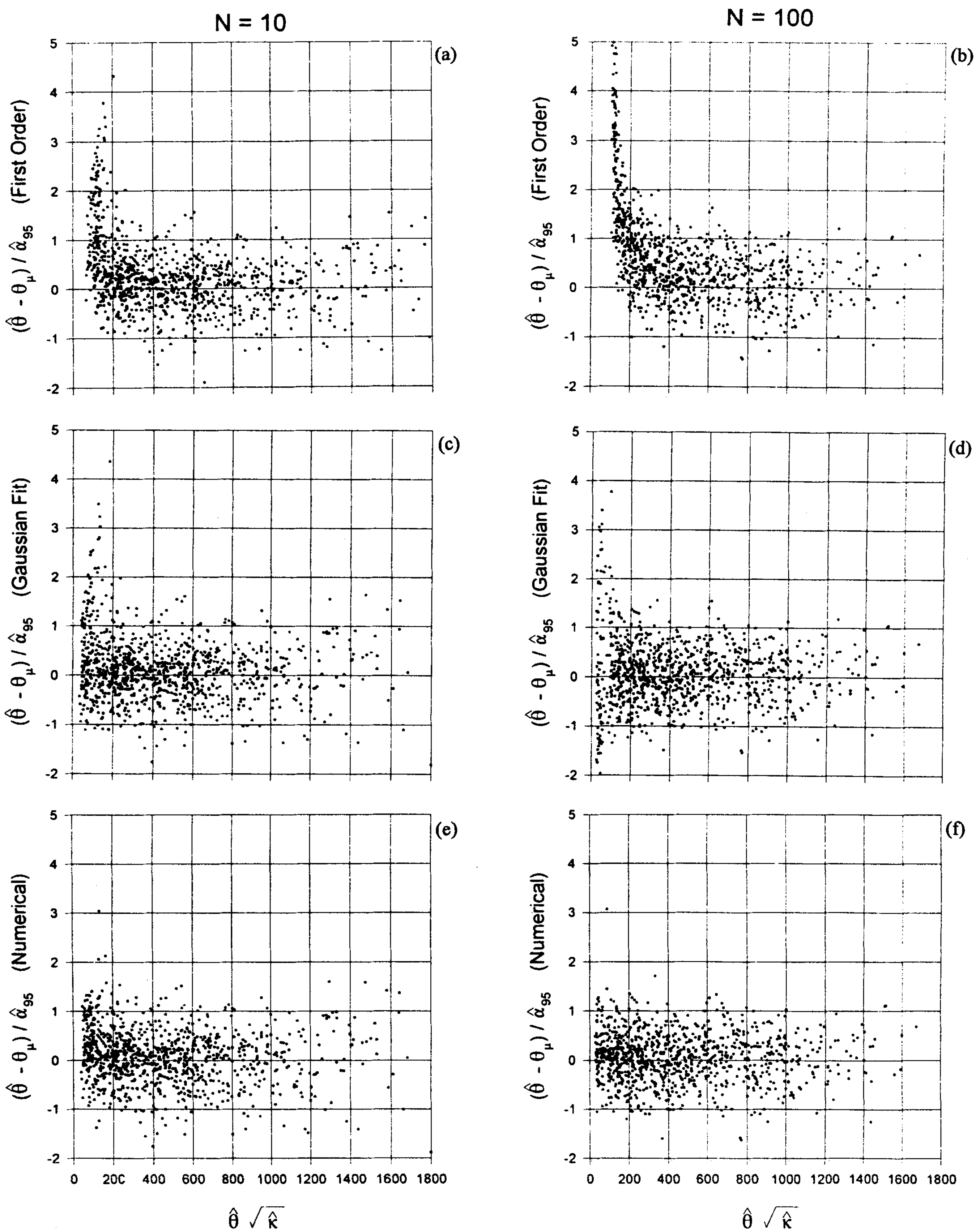
We have three methods of estimating the mean inclination and its associated 95 per cent confidence interval. The first-order estimate [eqs (10)–(12)] is merely the arithmetic mean plus or minus roughly twice the standard error. A more accurate method is to locate the maximum of the log-likelihood function $\ell(\hat{\theta}, \hat{\kappa})$ (16) by iteratively cycling between (19) and (20). The likelihood function of θ (21) is usually fitted well by a normal curve with mean $\hat{\theta}_2$ and variance $1/N\hat{\kappa}$, so the 95 per cent confidence interval is $\hat{\theta}_2 \pm [(1.960/\sqrt{N\hat{\kappa}})180°/\pi]$. Finally, when the inclination is too steep or the precision too poor for the normal curve approximation, the maximum likelihood inclination and confidence interval can be numerically determined from (21), (22) and (23).

It is certainly preferable to use a mean and standard error when possible, so one needs a simple criterion to decide which method to use. Problems arise with the simple estimates when some of the data are close to the vertical. Distortions due to spherical geometry only become important near the vertical, and there is the possiblility that some directions overshoot the vertical.

The dispersion of a Fisher-distributed sample is proportional to $1/\sqrt{\kappa}$ (for $\kappa > 3$). For example, 50 per cent of measurements should be more than $67.5°/\sqrt{\kappa}$ from the mean, but only 5 per cent should be more than $140°/\sqrt{\kappa}$ from the mean (Watson & Irving 1957). If the co-inclination is much greater than the dispersion then the probability of some of the measurements being close to vertical will be very small. A useful measure of this effect is $\hat{\theta}\sqrt{\hat{\kappa}}$, which is similar to a mean over a standard deviation. When $\bar{\theta}\sqrt{\kappa^*}$ or $\hat{\theta}_2\sqrt{\hat{\kappa}}$ is high enough, there is no need to integrate the likelihood function numerically. Of course, when $\hat{\theta}_2 > 90°$ the value to be considered is $(180° - \hat{\theta}_2)\sqrt{\hat{\kappa}}$. Intuitively, we expect the normal approximation to be accurate when the mean inclination is greater than about two to three standard deviations from the vertical.

As we are not privy to the true θ and κ we must make do with their estimates. Rather than attempt a theoretical estimate of the critical value of $\theta\sqrt{\kappa}$ above which a given estimation method is valid, we performed numerical simulations of Fisher-distributed samples with various θ_μ, κ and N to get empirical estimates and simultaneously to verify our methods. The declinations of the directions were ignored for the inclination-only estimates.

In Fig. 3 we show a suite of graphs summarizing the results from 1000 trials with $N = 10$ and $N = 100$. θ_μ was chosen between 0° and 90° and κ between 3 and 300. For each of the three methods, we plot, as a function of $\hat{\theta}\sqrt{\hat{\kappa}}$, the deviation of the estimate from the true inclination normalized by α_{95},



N = 10
N = 100
(a)
(b)
(c)
(d)
(e)
(f)
$(\hat{\theta} - \theta_\mu) / \hat{\alpha}_{95}$ (First Order)
$(\hat{\theta} - \theta_\mu) / \hat{\alpha}_{95}$ (Gaussian Fit)
$(\hat{\theta} - \theta_\mu) / \hat{\alpha}_{95}$ (Numerical)
5
4
3
2
1
0
-1
-2
0 200 400 600 800 1000 1200 1400 1600 1800
$\hat{\theta}\sqrt{\hat{\kappa}}$

$(\hat{\theta} - \theta_\mu)/\alpha_{95}$. One expects this value to be symmetrically distributed about 0 and within ± 1, 95 per cent of the time. If the normalized deviations tend to be positive, then the inclination estimate is biased towards shallow directions. If too many lie outside ± 1 then the confidence interval is underestimated.

The top graphs (Figs 3a and b) are for the first-order estimates, those in the middle (Figs 3c and d) are for estimates using a Gaussian fit of the inclination likelihood function, and those at the bottom (Figs 3e and f) show results for numerical analysis of the likelihood function. Note that for the last method, the confidence interval is not symmetric about the mean, so the appropriate value of α_{95}, depending on the sign of the deviation, had to be used. When $\hat{\theta} - \theta_\mu > 0$, we plot $(\hat{\theta} - \theta_\mu)/\alpha_{95}^+$, or otherwise, $(\hat{\theta} - \theta_\mu)/\alpha_{95}^-$.

The most important conclusion to be drawn from the numerical simulations is that the numerical analysis of the likelihood function (Figs 3e and f) gives good results regardless of θ_μ, κ and N. The normalized deviation is within ± 1, 94 per cent of the time. The method can be recommended for all inclination-only situations. For large $\hat{\theta}\sqrt{\hat{\kappa}}$, fitting the likelihood function with a Gaussian curve gives indistinguishable results. Empirically we find that the Gaussian estimates are acceptable when $\hat{\theta}\sqrt{\hat{\kappa}} > 200°$ for $N < 30$, and $\hat{\theta}\sqrt{\hat{\kappa}} > 150°$ for larger N. The normalized deviations for the arithmetic mean are mostly affected by the bias problem, but for $\hat{\theta}\sqrt{\hat{\kappa}} > 400°$, this simple estimate is adequate.

To summarize, the first-order estimate of the inclination and confidence interval comes from the arithmetic mean and standard deviation of the data, (10) and (12). Using these estimates, if $\hat{\theta}\sqrt{\hat{\kappa}} > 400°$ then there is no advantage in using more complicated procedures. More accurate estimates come from maximizing the log-likelihood function (16). The marginal likelihood function in θ alone (21) is approximately Gaussian in shape, leading to a simple expression for α_{95}. When the value of $\hat{\theta}\sqrt{\hat{\kappa}}$ is under 200° (or under 150° for $N \geq 30$), it is preferable to estimate the inclination and asymmetric confidence interval using numerical integration of the single-parameter likelihood function (21).

BLOCK-ROTATION FISHER ANALYSIS

Sometimes a sample collection consists of several groups, each from a separate rigid block. The blocks may have suffered an unknown vertical-axis rotation. Within each block the relative declination information is usable and should not be discarded. A block could be an unbroken segment of a core with several subsamples, or it could be an unfaulted locality with several sampling sites. Consider, for example, the study of the mid-Cretaceous Sverdrup Basin volcanics from Axel Heiberg Island in the Canadian Arctic (Wynne, Irving & Osadetz 1988). Accurate determination of the palaeomagnetic inclination is necessary to discriminate between two opposing models of the separation of Greenland from North America. The inclination is near vertical and the inclination-only method cannot resolve the best inclination well. Sampling localities showed no evidence of internal deformation and can be considered rigid blocks. The declination scatter between blocks is far greater that within blocks (Wynne *et al.* 1988, Fig. 17).

Wynne *et al.* (1988) used a method first introduced by Monger & Irving (1980), which they called 'Modified Fisher' analysis; however, we prefer the more descriptive title 'block-rotation Fisher' (BRF) analysis. Fisher averages are determined for each block, and then each block is rotated about the vertical such that the mean declinations of the blocks are coincident. A global average of these rotated site means is then made.

While the block-rotation method was introduced on intuitive grounds, we will now show that it is essentially identical to the maximum likelihood approach. Earlier we sought only estimates of θ_μ and κ, but now we must also seek estimates of the m block azimuths, $\phi_j, j = 1, \ldots, m$. Similarly to in eq. (14), the Bayesian likelihood function is given by

$$L_{\text{Bayes}} = \left(\frac{\sin\theta_\mu}{\kappa}\right)\left(\frac{\kappa}{2\sinh\kappa}\right)^N \prod_{j=1}^{m}\prod_{i=1}^{n_j} \sin\theta_{ji} \times \exp(\kappa[\cos\theta_\mu\cos\theta_{ji} + \sin\theta_\mu\sin\theta_{ji}\cos(\phi_j - \phi_{ji})]), \tag{24}$$

where $N = \Sigma_{j=1}^m n_j$.

The maximum likelihood solution for each ϕ_j is particularly simple because they are not dependent on each other, nor on θ_μ and κ. Differentiating the logarithm of the likelihood function with respect to ϕ_j and setting it equal to zero, we get

$$\tan\hat{\phi}_j = \frac{Y_j}{X_j}, \tag{25}$$

with $X_j = \Sigma_{i=1}^{n_j}\sin\theta_{ji}\cos\phi_{ji}$ and $Y_j = \Sigma_{i=1}^{n_j}\sin\theta_{ji}\sin\phi_{ji}$. This result is the Fisher estimate of the declination of each block. One could attempt the same procedure for estimating θ_μ and κ, but the results would be biased. The ϕ_js are nuisance parameters in this situation, since their values are irrelevant when one is interested in estimating the mean inclination of the sample. More intuitively, if one rotates all the blocks such that their mean declinations are aligned, those blocks that give poor estimates of the declination will pull the mean inclination shallow. It is necessary to integrate out the ϕ_js (i.e. $\int_0^{2\pi} d\phi_j$) to produce the two-parameter likelihood function

$$L_2 = \left(\frac{\sin\theta_\mu}{\kappa}\right)\left(\frac{\kappa}{\sinh\kappa}\right)^N \prod_{j=1}^{m} \exp(\kappa\cos\theta_\mu Z_j) I_0(\kappa\sin\theta_\mu H_j), \tag{26}$$

where $Z_j = \Sigma_{i=1}^{n_j}\cos\theta_{ji}$, $H_j^2 = X_j^2 + Y_j^2$, and constant factors have been taken out. When there is only one observation per block, the block-rotation Fisher analysis reduces to the inclination-only analysis outlined above.

The determination of $\hat{\theta}$ and its corresponding confidence interval is carried out the same way for the inclination-only

Figure 3. Plots of normalized deviations $[(\hat{\theta} - \theta_\mu)/\alpha_{95}]$ as a function of the distance from the vertical, parametrized as $\hat{\theta}\sqrt{\hat{\kappa}}$, for 1000 numerical trials with θ_μ chosen between 0° and 90° and κ chosen between 3 and 300, for $N = 10$ or 100. One expects normalized deviations to be symmetrically distributed about 0 and within ± 1 95 per cent of the time. (a), (b) Note that the first-order estimate (the arithmetic mean) gives adequate results when $\hat{\theta}\sqrt{\hat{\kappa}} > 400°$. Below that value, the estimate is unacceptably biased shallow. (c), (d) The Gaussian fit of the likelihood function allows adequate estimates for $\hat{\theta}\sqrt{\hat{\kappa}}$ down to 200° (150° for $N \geq 30$), but below that the asymmetric shape of the likelihood function must be taken into account. (e), (f) Numerical estimation of the mean inclination and its confidence interval is suitable regardless of how steep or dispersed the data are.

problem. We iteratively solve

$$\tan \hat{\theta} = \left(\frac{1}{N} \sum_{j=1}^{m} H_j \frac{I_0'}{I_0}(\hat{\kappa} \sin \hat{\theta}\, H_j) + \frac{1}{N\hat{\kappa} \sin \hat{\theta}} \right) \Big/ \left(\frac{1}{N} \sum_{j=1}^{m} Z_j \right), \tag{27}$$

and

$$\hat{\kappa} = (N-1) \Big/ \left(N - \sum_{j=1}^{m} \left\{ \cos \hat{\theta}\, Z_j + \sin \hat{\theta}\, H_j \frac{I_0'}{I_0}(\hat{\kappa} \sin \hat{\theta}\, H_j) \right\} \right). \tag{28}$$

When $\kappa \sin \theta$ is large, these equations become $\tan \hat{\theta} = \Sigma_{j=1}^{m} H_j / \Sigma_{j=1}^{m} Z_j$ and $\hat{\kappa} = (N-1)/(N-R)$, where $R^2 = \Sigma_{j=1}^{m} (H_j^2 + Z_j^2)$. These simplified equations are what one would find by rotating each block to a common mean declination and then determining $\hat{\theta}$ and $\hat{\kappa}$ by standard Fisher analysis. The $\hat{\kappa}$ estimate assumes a value >3. The bias in $\hat{\kappa}$ can be corrected by taking the number of degrees of freedom into account. There are $2N$ observations (declination and inclination of each site) while we are estimating $2m+2$ parameters. Watson (1956) demonstrated that the numerator of the $\hat{\kappa}$ equation should be half the degrees of freedom, $N-(m+1)/2$. Our simulations show that this modified equation is useful, regardless of N, m, or $\kappa \sin \theta$. On the other hand, iterative solution is not time consuming, and also corrects the inclination bias.

When block-rotation Fisher analysis can be applied (i.e. where sites are distributed among rigid blocks), the two-parameter likelihood function has a simpler form than that for the same data set using inclination-only analysis. Fig. 4 was produced using the Sverdrup Basin data (Table 1). Fig. 4(a) shows contours of log likelihood when each site is allowed to rotate freely about a vertical axis (inclination-only analysis) and Fig. 4(c) shows the contours for the same data, except that groups of sites are constrained on geological grounds to rotate together. Note that these contours have a more elliptical shape, indicating that the Gaussian approximation fits the function quite well. The marginal likelihood function using block-rotation Fisher analysis (Fig. 4d) is tighter and more symmetrical than that using inclination-only analysis (Fig. 4b). Our simulations indicate that it is rarely necessary to perform a numerical integration of the BRF likelihood function to obtain a reasonable confidence interval. The marginal likelihood of θ alone is approximately Gaussian with variance $1/N\hat{\kappa}$, so the 95 per cent confidence interval is $\hat{\theta} \pm [(1.960/\sqrt{N\hat{\kappa}})180°/\pi]$.

Unlike the inclination-only situation, it is impossible to offer a simple $\theta\sqrt{\kappa}$ rule that is applicable to all BRF situations. Through our simulations we find that the $\hat{\theta}$ and α_{95} determined through the Gaussian fit is usually almost identical to that determined by numerical integration. When $\alpha_{95} \approx \hat{\theta}$, the maximum of the two-parameter likelihood function (26) still gives practically the same $\hat{\theta}$ as with numerical fitting of the marginal likelihood function, but the Gaussian fit of α_{95} is slightly

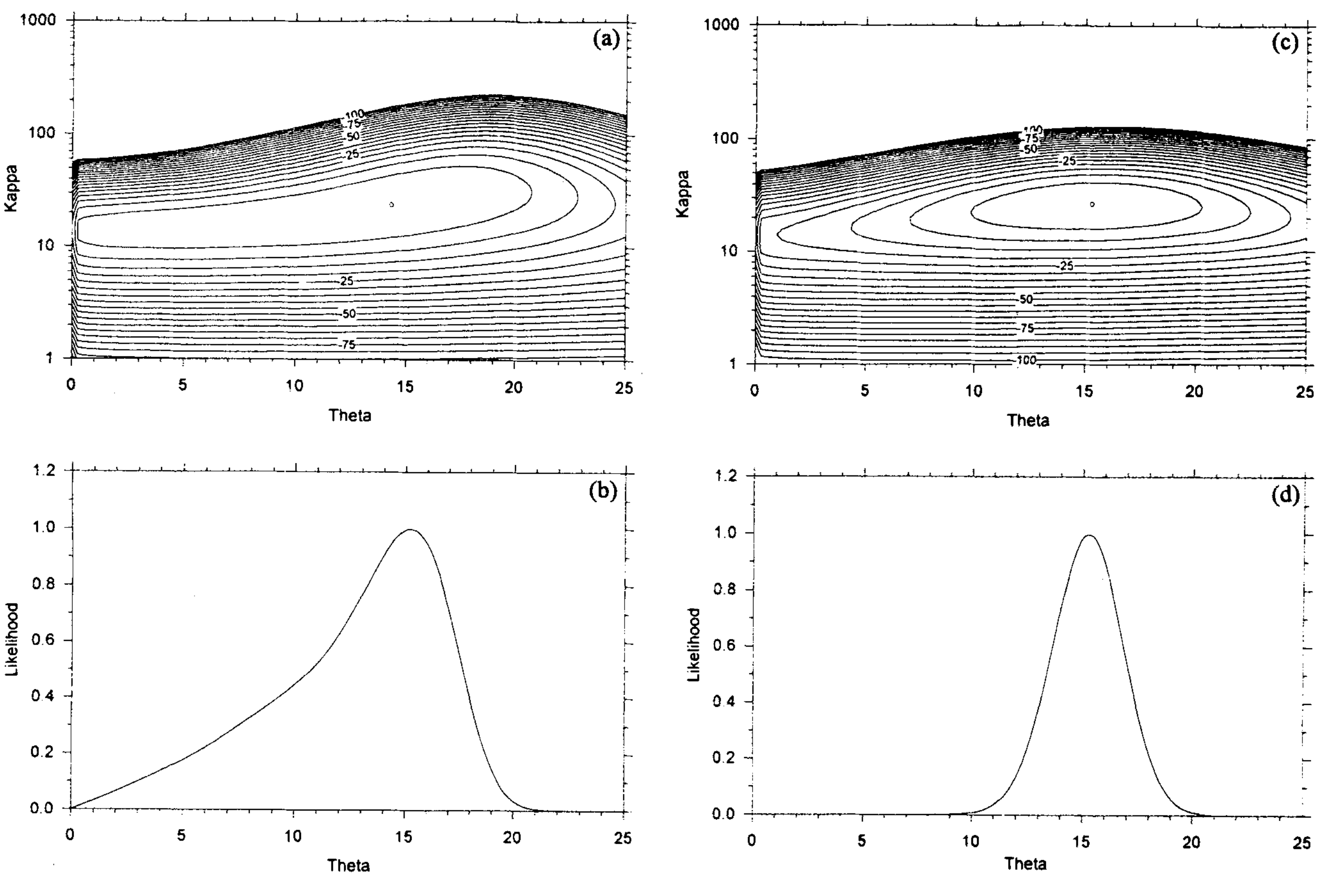


Figure 4. Plots of the likelihood function for the Sverdrup Basin data set (Table 1; Wynne *et al.* 1988) using (a), (b) inclination-only analysis and (c), (d) block-rotation Fisher analysis. Note that the contours on (c) are far more elliptical in shape than in (a), and thus the Gaussian fit is more appropriate for the marginal likelihood function (d). The 95 per cent confidence interval is reduced from 15.7° using inclination-only analysis to 5.8° using block-rotation Fisher analysis.

Table 1. Site-mean directions (after horizontal-axis tectonic correction) of Sverdrup Basin volcanics (summarizing Wynne *et al.* 1988, Tables 1, 2 and 3). This data set is composed of all sites with extrusive lithologies, $k > 10$ and $\alpha_{95} < 20°$. Sites are given as normal-polarity directions. Each rigid block is denoted by a distinct letter label.

Site	Dec.	Inc.	Block	Site	Dec.	Inc.	Block
RUE41	305	59	A	AXV30	327	64	K
RUE44	201	87	B	AXV31	322	74	K
RUE45	046	75	B	AXV32	231	63	K
AXW04	315	75	C	AXV34	218	68	K
AXW08-09	235	52	C	AXV35	239	68	K
AXW12	202	59	C	AXV36	228	55	K
AXW13-14	311	72	C	AXV37	246	58	K
AXW18	206	81	D	AXA05	036	74	L
AXB46	300	75	E	AXA06	002	80	L
AXB50	257	82	F	AXA01	344	84	L
AXB49	202	75	G	AXA02	206	88	M
AXB32	254	79	G	AXA03	024	81	M
AXB31-33	259	66	G	AXA04	097	79	M
AXB43	213	85	H	AXS04	214	77	N
AXB42	145	88	H	AXS02	211	66	N
AXB41	160	78	H	AXS07	203	76	N
AXB40	300	77	H	AXS09	264	62	N
AXB38	115	81	H	AXS15	316	55	O
AXB37	154	75	H	AXS14	321	64	O
AXB35	185	70	H	AXS13	349	62	O
AXB01	178	64	I	AXS12	328	62	O
AXB02	150	79	I	AXS10-11	340	67	O
AXV10	245	68	J	AXS25	228	78	P
AXV01	155	69	J	AXS24	248	51	P
AXV02	148	75	J	AXS22	231	58	P
AXV03	289	71	J	AXS21	246	57	P
AXV04	165	58	J				
AXV05	208	66	J				
AXS09	236	67	J				

overestimated. We suggest that when $\hat{\theta} - \alpha_{95} < 2°$, numerical integration should be used. This situation will rarely occur in palaeomagnetic studies.

Inclination-only analysis of the Sverdrup Basin sites of Wynne *et al.* (1988) gives $\hat{I} = 74.9^{+11.0}_{-4.7}$, $\hat{\kappa} = 21.5$, but $\hat{\theta}\sqrt{\hat{\kappa}} = 69°$, which is quite low and definitely in the range of high spherical distortion. Applying the block-rotation Fisher method, we get $\hat{I} = 74.7° \pm 2.9°$, $\hat{\kappa} = 26.6$. The 95 per cent confidence range dropped from 15.7° using inclination-only analysis to 5.8° with BRF analysis, a very substantial improvement in accuracy.

INCLINATION-ONLY FOLD TEST

Even when declination information is unavailable, it is often desirable to determine the relative timing of magnetic remanence and tectonic deformation using some formulation of the palaeomagnetic fold test. Most fold-test formulations are based on the conjecture that the remanence directions should be tightly clustered when the correct tectonic rotations have been applied and that any other rotations will produce more dispersed distributions. Unfortunately, in situations where declination information is lost, complete undoing of tectonic rotations is impossible.

Fold-test formulations have traditionally been posed in terms of hypothesis tests. In our recent paper on the fold test (Watson & Enkin 1993) we proposed that it is better to restate the problem as one of parameter estimation. In particular, we argued that finding the degree of untilting that gives minimum dispersion is the best way to determine whether the magnetization is pre-tilting, post-tilting or in some intermediate case (due to any of a numer of causes). The common plot of Fisher's precision parameter as a function of the degree of untilting will usually have a single maximum, and if that maximum is close to 100 per cent untilting then one can say that the remanence was acquired before the beds were tilted.

To find a 95 per cent confidence interval for the optimal degree of untilting we recommended using parametric resampling at the site level. This method can be adapted to the inclination-only problem. The n, κ and mean direction before tectonic correction of each site are used to choose a new Fisher-distributed data set [using, for example, the algorithm of Fisher, Lewis & Embleton (1987) section 3.6.2). These simulated data are then used to define new site means. Using these simulated site means, inclination-only analysis is applied and $\hat{\kappa}$ is determined as a function of degree of untilting using the measured bedding orientations. The percentage of untilting, γ_m, that produces the maximum precision is stored in a list. This procedure is repeated many times and then the list of γ_ms is sorted. The best estimate of optimum untilting is the median (50 percentile), while the lower and upper 95 per cent confidence limits are the 2.5 and 97.5 percentile values. If the optimum untilting interval includes 100 per cent untilting and excludes 0 per cent untilting, the fold test is considered positive.

As an example, consider the study of the lower Jurassic Morrison Formation by Van Fossen & Kent (1992). The distribution of site-mean directions is streaked for any degree of untilting, and the region was sufficiently disturbed to suggest that vertical-axis rotations may have occurred. They attempted to perform the fold test by using McFadden & Reid's (1982) inclination-only estimate of κ for pre- and post-tilting geometries, and then applying McElhinny's (1964) k-ratio fold-test formulation. However, the k-ratio formulation does not actually test what is purports to test (McFadden & Jones 1981), and its use should be abandoned.

Fig. 5 shows results using our method. Progressive untilting curves of the Fisher precision parameter vary in magnitude much more than is typical of full-data simulations, but the maxima are all well defined and in a restricted range (20 curves are shown in the figure). After 1000 resampling trials, we find that the optimum degree of untilting is 79.6 per cent with lower and upper confidence limits of 53.0 per cent and 107.9 per cent. As this range includes full untilting (100 per cent) but not the *in situ* (0 per cent) case, the fold test is positive, as claimed by Van Fossen & Kent.

CONCLUSION

Correctly determining the mean inclination of a palaeomagnetic data set is important to many studies. In this paper we have presented statistical tools for estimating the inclination and its confidence interval that are applicable to most palaeomagnetic situations, including steep or highly dispersed data. Furthermore, we give a simple criterion to help the practitioner decide which method is preferable. Our methods have been tested using real data and by extensive numerical simulation.

To estimate the mean inclination of a data set, the first step is to calculate the arithmetic mean, $\bar{I}$, and variance (the standard deviation squared). The reciprocal of the variance is an estimate of Fisher's precision parameter, κ^*. The first-order estimate of

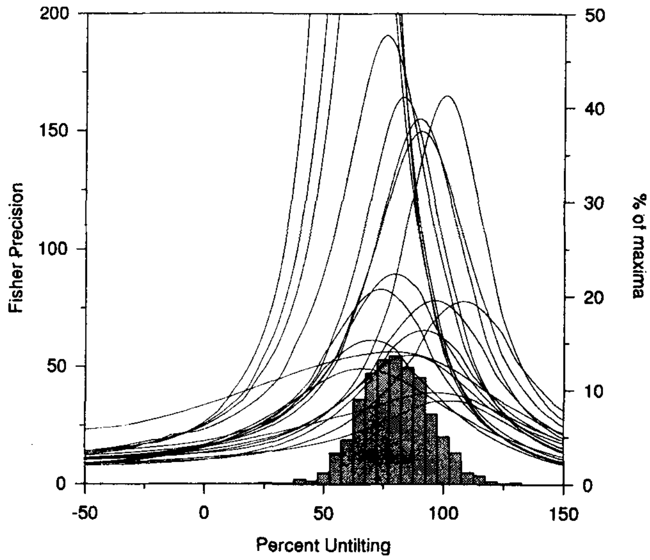


Figure 5. Application of the fold-test formulation of Watson & Enkin (1993) to inclination-only data, using Morrison Formation data from Van Fossen & Kent (1992). The continuous curves show 20 examples of parametric resamplings of the original data, showing precision as a function of the degree of untilting. The histogram shows the distribution of the maxima from 1000 such curves. Since the central 95 per cent range of this distrubution includes full untilting (100 per cent) but not the *in situ* case (0 per cent) the fold test is positive.

the 95 per cent confidence interval is $\bar{I}\pm$ approximately twice the standard error (eq. 12).

To determine whether these first-order estimates are valid, calculate $\bar{\theta}\sqrt{\kappa^*}$, where $\theta = 90^\circ - |I|$ is the co-inclination. $\theta\sqrt{\kappa}$ is similar to a mean over its standard deviation, so a high $\theta\sqrt{\kappa}$ means the dispersion of the data is much smaller than the distance from the vertical, and the probability of some members of the distribution being close to the vertical is small. Through numerical simulations, we have determined that there is no advantage to performing more complicated estimates of the inclination when $\bar{\theta}\sqrt{\kappa^*} > 400^\circ$. For example, if the mean inclination is 30°, this simple estimate is acceptable as long as κ^* is greater than 44. But if $\bar{I} = 60^\circ$ then the result will be biased shallow unless $\kappa^* > 178$, which is seldom the case in palaeomagnetic studies.

If $\bar{\theta}\sqrt{\kappa^*} < 400^\circ$, spherical distortion will bias the arithmetic mean of the inclinations towards shallow values. In this case, it is better to determine the inclination (or co-inclination, $\hat{\theta}_2$) and precision ($\hat{\kappa}$) that maximize the Bayesian likelihood (14) by iteratively cycling between eqs (19) and (20). If $\hat{\theta}_2\sqrt{\hat{\kappa}} > 200^\circ$ ($> 150^\circ$ if $N > 30$), the correct estimate is given by $\hat{I} = 90^\circ - \hat{\theta}_2 \pm [(1.960/\sqrt{N\hat{\kappa}})180^\circ/\pi]$. If the data are so steep or dispersed that $\hat{\theta}_2\sqrt{\hat{\kappa}} < 200^\circ$ (or $< 150^\circ$ for large N), then the mean and asymmetric confidence interval must be determined numerically using (21), (22) and (23).

In situations where the data come from sites that belong to rigid blocks that have suffered relative vertical-axis rotations, the declination information within each block can be used to provide better inclination estimates using a method we call block-rotation Fisher analysis. The co-inclination and precision are determined by cycling iteratively between eqs (27) and (28). The confidence intervals are determined in the same way as for the inclination-only problem, but will be smaller because the declination information has been utilized.

To perform a fold test using inclination data, we demonstrate the use of our parameter-estimation formulation (Watson & Enkin 1993). If the degree of untilting that produces the highest precision is around 100 per cent, but significantly different from 0 per cent, then the fold test can be considered positive.

It is our hope that these statistical tools will find application in many palaeomagnetic studies. All the methods described in this paper are included in the package of palaeomagnetism analysis programs available from the first author.

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